ASIAN JOURNAL OF STATISTICS AND APPLICATIONS

VOL NO. 1, ISSUE 2 ,PAGE NO. 1-33, YEAR 2024

Odds One Parameter Polynomial Exponential - G Family of Distributions

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ARTICLE HISTORY
Online Published December 27, 2024

ABSTRACT
To model lifetime data, the study suggests a new class of distributions termed the Odds One Parameter Polynomial Exponential-G family of distributions. Incomplete moments, Mean deviations, Lorenz and Bonferroni curves, Moments of residual and reversed residual life, Shape, Quantile function, Entropy, Order Statistics, Stress-Strength reliability, and other structural and reliability properties are discussed. Estimation of the parameters involved has been described by the maximum likelihood method. The results of simulation studies have been reported. Four data sets have been analyzed to demonstrate the applicability of the class of distributions.

KEYWORDS
Mixture distribution, Odds function, Reliability properties, Structural properties, T-X family of distributions.

1. Introduction

Data analytics is now a crucial tool for the growth and development of any corporate organization, whether it is the manufacturing or service industries. As data patterns become more complex over time, traditional probability distributions cannot accurately depict the proper distributional structure of quantitative data. For analyzing intricately structured industrial data, transformed distributions using classical probability distributions as bases could be used efficiently.
By McDonald (1984), Azzalini (1985), Marshall and Olkin (1997), and others, several generic techniques for generating a new family of distributions are proposed, and their features and statistical inference are investigated. The work of Eugene et al. (2002) at the beginning of the century saw the development of the beta-generated family of distributions. By substituting the Kumaraswamy distribution for the beta distribution, Jones (2009) and Cordeiro and deCastro (2011) followed the extension of the beta-generated family of distributions.
A generalized family of distributions known as the T-X (also known as Transformed-Transformer) family was proposed by Alzaatreh et al. (2013), whose cumulative dis-

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tribution function (cdf) is given by

$$F(v;\theta) = \int_a^{W[G(v)]} r(t)dt, \tag{1}$$

where, the random variable $T \in [a, b]$, for $-\infty < a, b < \infty$ and $W[G(v)]$ be a function of the cdf $G(v)$ so that $W[G(v)]$ satisfies the following conditions:

(i) $W[G(v)] \in [a, b]$,
(ii) $W[G(v)]$ is differentiable and monotonically non-decreasing,
(iii) $W[G(v)] \to a$ as $v \to -\infty$ and $W[G(v)] \to b$ as $v \to \infty$.

For studying the life of any product/service, the widely used probability law is the exponential distribution because of its simple and interesting character. Moreover, statistical inference is relatively easy and comprehensive for practitioners and scientists. Due to its flexibility in some aspects, such as the mode, mean deviation, moments, skewness and kurtosis measurements, failure rate and mean residual life, entropies, etc., the Lindley distribution has recently gained favour over the exponential distribution. The mixture of an exponential distribution and a gamma distribution with shape parameter 2 results in the Lindley distribution.
The probability density function (pdf) of the Lindley distribution [see, Lindley (1958)] is given by

$$f(v;\lambda) = \frac{\lambda^2}{1+\lambda}(1+v)e^{-\lambda v}, \qquad \lambda, v > 0.$$

A host of authors has made generalizations of this model from different angles. Zakerzadeh and Dolati(2010), Bakouch et al. (2012), Shanker et al. (2013), Elbatal et al. (2013), Ghitany et al. (2013), Singh et al. (2014), Abouamoh et al. (2015), among others, are worth mentioning. Bouchahed and Zeghdoudi (2018) have proposed a new and unified approach to generalizing Lindley's distribution. The generalized distribution may be called a one-parameter polynomial exponential (OPPE) family of distributions. Some structural properties like moments, skewness, kurtosis, median, mean deviations, Lorenz curve, entropies, and limiting distribution of extreme order statistics; reliability properties like reliability function, hazard rate, stress-strength reliability, stochastic ordering; and estimation methods like the method of moment and maximum likelihood have been investigated. The pdf of the random variable X belonging to OPPE family can be written as

$$f_V(v;\lambda) = \frac{\sum_{k=0}^{s} a_k v^k e^{-\lambda v}}{\sum_{k=0}^{s} a_k \frac{\Gamma(k+1)}{\lambda^{k+1}}}, \qquad \lambda, v > 0. \tag{2}$$

The distribution can also be written as

$$f_V(v;\lambda) = h(\lambda)\sum_{k=0}^{s} a_k v^k e^{-\lambda v} = h(\lambda)\sum_{k=0}^{s} a_k \frac{\Gamma(k+1)}{\lambda^{k+1}} f_{GA}(v;k+1,\lambda)$$

where, $h(\lambda) = \frac{1}{\sum_{k=0}^{s} a_k \frac{\Gamma(k+1)}{\lambda^{k+1}}}$, and $f_{GA}(v;k+1,\lambda)$ is a gamma pdf with shape parameter $(k+1)$ and scale parameter λ, and a_k's are non-negative constants. A mixture of $(s+1)$ gamma distributions constitutes the distribution.

The exponential (for $s = 0$, $a_0 = 1$), Lindley (for $s = 1$, $a_0 = 1$, $a_1 = 1$), Akash (for $s = 2$, $a_0 = 1$, $a_1 = 0$, $a_2 = 1$) [c.f. Shankar(2015a)], Aradhana (for $s = 2$, $a_0 = 1$, $a_1 = 2$, $a_2 = 1$) [c.f. Shankar(2016a)], Sujatha (for $s = 2$, $a_0 = 1$, $a_1 = 1$, $a_2 = 1$) [c.f. Shankar(2016b)], length-biased Lindley (for $s = 2$, $a_0 = 0$, $a_1 = 1$, $a_2 = 1$) [c.f. Ayesha(2017)], Amarendra (for $s = 3$, $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$) [c.f. Shankar(2016c)], Devya (for $s = 4$, $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$) [c.f. Shankar(2016d)], Shambhu (for $s = 5$, $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$) distribution [c.f. Shankar(2016e)] are special cases.

In the article, we propose a new wider class of continuous distributions called the Odds One-Parameter Polynomial Exponential - G family by taking $W[G(v)] = \frac{G(v;\xi)}{1-G(v;\xi)}$, the odds function of cdf and $r(t) = h(\lambda)\sum_{k=0}^{s} a_k t^k e^{-\lambda t}, t > 0, \lambda > 0$, the generator. Here $G(v;\xi)$ is a baseline cdf, which depends on a parameter vector ξ and $\bar{G}(v;\xi) = 1 - G(v;\xi)$ is the survival function. $\Gamma(p,v) = \int_v^\infty w^{p-1}e^{-w}dw$, the upper incomplete gamma function and $\gamma(p,v) = \int_0^v w^{p-1}e^{-w}dw$, the lower incomplete gamma function for $v \geq 0, p > 0$ respectively are the notations used in the article. $\Gamma^{(j)}(p,v) = \int_v^\infty (\ln w)^j w^{p-1}e^{-w}dw$ and $\gamma^{(j)}(p,v) = \int_0^v (\ln w)^j w^{p-1}e^{-w}dw$, for $v \geq 0, p > 0$ respectively are the j^{th} derivative with respect to p. Maiti and Pramanik (2015, 2016a, 2016b) developed the Odds Generalized Exponential-Exponential, Exponential-Uniform, and Exponential-Pareto distributions. The properties were studied and illustrated with applications. Maiti and Pramanik (2018) also developed a family of distributions known as the Odds xgamma-G family, using the xgamma distribution as the generator.

The cdf of Odds OPPE - G family of distributions is given by

$$\begin{aligned} F(v;\lambda,\xi) &= \int_0^{\frac{G(v;\xi)}{1-G(v;\xi)}} h(\lambda)\sum_{k=0}^{s} a_k t^k e^{-\lambda t}dt \\ &= 1 - h(\lambda)\sum_{k=0}^{s} a_k \frac{\Gamma\left(k+1, \lambda\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right)}{\lambda^{k+1}} \end{aligned} \tag{3}$$

where, $h(\lambda) = \frac{1}{\sum_{k=0}^{s} a_k \frac{\Gamma(k+1)}{\lambda^{k+1}}}$

The probability density function (pdf) of Odds OPPE - G family of distribution is

$$f(v;\lambda,\xi) = h(\lambda)\sum_{k=0}^{s} a_k \frac{g(v;\xi)}{[\bar{G}(v;\xi)]^2}\left[\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right]^k e^{-\lambda\left[\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right]}. \tag{4}$$

The survival function of Odds OPPE - G family of distributions is

$$S(v;\lambda,\xi) = h(\lambda)\sum_{k=0}^{s} a_k \frac{\Gamma\left(k+1, \lambda\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right)}{\lambda^{k+1}}. \tag{5}$$

Table 1. Distributions and corresponding $G(v;\xi)/\bar{G}(v;\xi)$ functions

Distribution	$G(v;\xi)/\bar{G}(v;\xi)$	ξ
Uniform$(0 < v < \theta)$	$v/(\theta - v)$	θ
Exponential$(v > 0)$	$e^{\lambda v} - 1$	λ
Weibull$(v > 0)$	$e^{\lambda v^{\gamma}} - 1$	(λ, γ)
Frechet$(v > 0)$	$(e^{\lambda v^{\gamma}} - 1)^{-1}$	(λ, γ)
Half-logistic$(v > 0)$	$(e^{v} - 1)/2$	ϕ
Power function$(0 < v < 1/\theta)$	$[(\theta v)^{-k} - 1]^{-1}$	(θ, k)
Pareto$(v \geq \theta)$	$(v/\theta)^{k} - 1$	(θ, k)
Burr XII$(v > 0)$	$[1 + (v/s)^{c}]^{k} - 1$	(s, k, c)
Log-logistic$(v > 0)$	$[1 + (v/s)^{c}] - 1$	(s, c)
Lomax$(v > 0)$	$[1 + (v/s)]^{k} - 1$	(s, k)
Gumbel$(-\infty < v < \infty)$	$[exp[exp(-(v - \mu)/\sigma)] - 1]^{-1}$	(μ, σ)
Kumaraswamy$(0 < v < 1)$	$(1 - v^{a})^{-b} - 1$	(a, b)
Normal$(-\infty < v < \infty)$	$\Phi((v - \mu)/\sigma)/(1 - \Phi((v - \mu)/\sigma))$	(μ, σ)

The hazard rate function of Odds OPPE - G family of distribution is

$$\begin{aligned} h(t;\lambda,\xi) &= \frac{f(t;\lambda,\xi)}{S(t;\lambda,\xi)} \\ &= \frac{\sum_{k=0}^{s} a_k \frac{g(t;\xi)}{[\bar{G}(t;\xi)]^2} \left[\frac{G(t;\xi)}{\bar{G}(t;\xi)}\right]^k e^{-\lambda\left[\frac{G(t;\xi)}{\bar{G}(t;\xi)}\right]}}{\sum_{k=0}^{s} a_k \frac{\Gamma\left(k+1,\lambda\frac{G(t;\xi)}{\bar{G}(t;\xi)}\right)}{\lambda^{k+1}}}. \end{aligned} \tag{6}$$

Odds function for different distributions and parameter vector ξ have been presented in Table 1.
The article is structured as follows. Section 2 discusses specific models assuming transformer distribution as Uniform, Exponential, and Burr XII. Section 3 discusses some mathematical properties such as Mixture Representation, Shape, Quantile function, Entropy, Order Statistics, Stress-Strength reliability, Incomplete moments, Mean deviations, Lorenz and Bonferroni curves, and Moments of residual and reversed residual life. In section 4, the maximum likelihood approach to parameter estimation was discussed. The simulation study methodology has been outlined, and section 5 shows simulation findings. In section 6, the fitting of the suggested model has been addressed and reported for four data sets. There are concluding remarks in section 7.

2. Some Special Models for Odds OPPE - G Family

In this section, some new special distributions, namely, Odds OPPE-Uniform, Odds OPPE-Exponential, Odds OPPE-Pareto, and Odds OPPE-Burr XII are introduced.

Table 2. Different forms of Odds OPPE - Uniform Distribution

Distribution Name	**Density Function**
Odds Exponential - Uniform[20]($s = 0,\ a_0 = 1$)	$f(v; \lambda, \theta) = \frac{\lambda\theta}{(\theta-v)^2} e^{-\frac{\lambda v}{\theta - v}}$
Odds Lindley - Uniform($s = 1,\ a_0 = 1,\ a_1 = 1$)	$f(v; \lambda, \theta) = \frac{\lambda^2}{1+\lambda} \frac{\theta^2}{(\theta-v)^3} e^{-\frac{\lambda v}{\theta - v}}$
Odds Akash - Uniform($s = 2,\ a_0 = 1,\ a_1 = 0,\ a_2 = 1$)	$f(v; \lambda, \theta) = \frac{\lambda^2}{1+\lambda} \frac{\theta^2}{(\theta-v)^3} e^{-\frac{\lambda v}{\theta - v}}$
Odds Aradhana - Uniform($s = 2,\ a_0 = 1,\ a_1 = 2,\ a_2 = 1$)	$f(v; \lambda, \theta) = \frac{\lambda^2}{1+\lambda} \frac{\theta^2}{(\theta-v)^3} e^{-\frac{\lambda v}{\theta - v}}$
Odds Sujatha - Uniform($s = 2,\ a_0 = 1,\ a_1 = 1,\ a_2 = 1$)	$f(v; \lambda, \theta) = \frac{\lambda^2}{1+\lambda} \frac{\theta^2}{(\theta-v)^3} e^{-\frac{\lambda v}{\theta - v}}$

2.1. *Odds OPPE - Uniform Distribution*

Consider the baseline distribution as uniform on the interval $(0, \theta),\ \theta > 0$ with the pdf and cdf, respectively

$$g(v; \theta) = \frac{1}{\theta}\ ; 0 < v < \theta < \infty,\ G(v, \theta) = \frac{v}{\theta}.$$

The cdf of Odds OPPE-Uniform distribution is obtained by substituting the cdf of uniform in (3) as follows

$$F(v; \lambda, \theta) \quad = \quad 1 - h(\lambda) \sum_{k=0}^{s} a_k \frac{\Gamma\left(k+1, \frac{\lambda v}{\theta - v}\right)}{\lambda^{k+1}}.$$

The corresponding pdf is given by

$$f(v; \lambda, \theta) \quad = \quad h(\lambda) \sum_{k=0}^{s} a_k \frac{\theta}{(\theta - v)^2} \left(\frac{v}{\theta - v}\right)^k e^{-\frac{\lambda v}{\theta - v}} \quad ; 0 < v < \theta < \infty, \lambda > 0.$$

A few particular forms of pdf of Odds OPPE-Uniform distribution has been listed in Table 2. Some shapes of pdf and survival function of the distribution are shown in Figure 1.

The survival and hazard rate functions are given respectively as follows:

$$S(v; \lambda, \theta) \quad = \quad h(\lambda) \sum_{k=0}^{s} a_k \frac{\Gamma\left(k+1, \frac{\lambda v}{\theta - v}\right)}{\lambda^{k+1}},$$

$$r(t; \lambda, \theta) \quad = \quad \frac{\sum_{k=0}^{s} a_k \frac{\theta}{(\theta-t)^2} \left(\frac{t}{\theta - t}\right)^k e^{-\frac{\lambda t}{\theta - t}}}{\sum_{k=0}^{s} a_k \frac{\Gamma\left(k+1, \frac{\lambda t}{\theta - t}\right)}{\lambda^{k+1}}}.$$

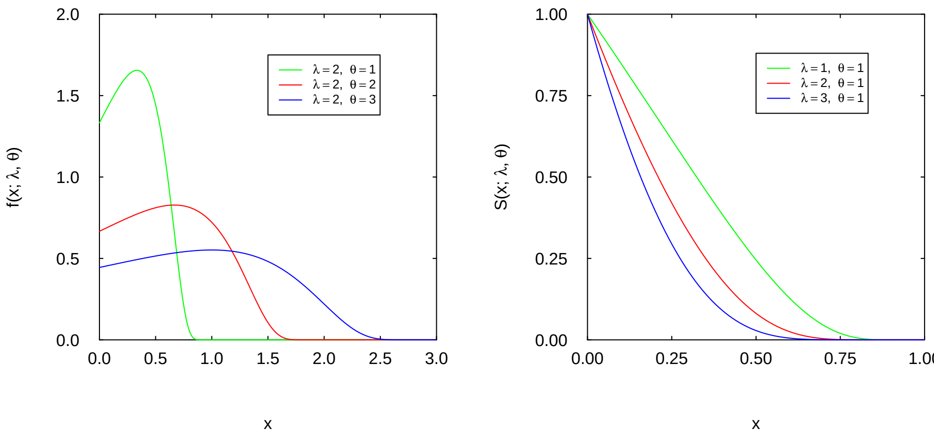

Figure 1. The pdf and survival function of Odds Lindley - Uniform distribution

2.2. *Odds OPPE - Exponential Distribution*

Let, the considered baseline distribution is Exponential with parameter $\theta > 0$ having the pdf and cdf, respectively

$$g(v;\theta) = \theta e^{-\theta v}\ ; 0 < v, \theta < \infty, \text{and } G(v,\theta) = 1 - e^{-\theta v}.$$

The cdf of Odds OPPE-Exponential distribution is obtained by substituting the cdf of Exponential in (3) as follows

$$F(v;\lambda,\theta) = 1 - h(\lambda)\sum_{k=0}^{s} a_k \frac{\Gamma\left(k+1, \lambda(e^{\theta v}-1)\right)}{\lambda^{k+1}}.$$

The corresponding pdf is given by

$$f(v;\lambda,\theta) = h(\lambda)\sum_{k=0}^{s} a_k \theta e^{\theta v}(e^{\theta v}-1)^k e^{-\lambda(e^{\theta v}-1)}\ ; 0 < v, \theta < \infty, \lambda > 0.$$

Some particular forms of pdf of Odds OPPE-Exponential distribution have been presented in Table 3. Some shapes of pdf and survival function of the distribution are shown in Figure 2.

The survival and hazard rate functions are as follows:

$$S(v;\lambda,\theta) = h(\lambda)\sum_{k=0}^{s} a_k \frac{\Gamma\left(k+1, \lambda(e^{\theta v}-1)\right)}{\lambda^{k+1}},$$

$$r(t;\lambda,\theta) = \frac{\sum_{k=0}^{s} a_k \theta e^{\theta t}(e^{\theta t}-1)^k e^{-\lambda(e^{\theta t}-1)}}{\sum_{k=0}^{s} a_k \frac{\Gamma(k+1,\lambda(e^{\theta t}-1))}{\lambda^{k+1}}}.$$

Table 3. Different forms of Odds OPPE - Exponential Distribution

Distribution Name	**Density Function**
Odds Exponential - Exponential[19]($s = 0,\ a_0 = 1$)	$f(v;\lambda,\theta) = \lambda\theta e^{\theta v} e^{-\lambda(e^{\theta v}-1)}$
Odds Lindley - Exponential($s = 1,\ a_0 = 1,\ a_1 = 1$)	$f(v;\lambda,\theta) = \frac{\lambda^2}{(1+\lambda)}\theta e^{2\theta v} e^{-\lambda(e^{\theta v}-1)}$
Odds Akash - Exponential($s = 2,\ a_0 = 1,\ a_1 = 0,\ a_2 = 1$)	$f(v;\lambda,\theta) = \frac{\lambda^3}{(2+\lambda^2)}\theta e^{\theta v}\left[1 + (e^{\theta v} - 1)^2\right] e^{-\lambda(e^{\theta v}-1)}$
Odds Aradhana - Exponential($s = 2,\ a_0 = 1,\ a_1 = 2,\ a_2 = 1$)	$f(v;\lambda,\theta) = \frac{\lambda^3}{(2+2\lambda+\lambda^2)}\theta e^{3\theta v} e^{-\lambda(e^{\theta v}-1)}$
Odds Sujatha - Exponential($s = 2,\ a_0 = 1,\ a_1 = 1,\ a_2 = 1$)	$f(v;\lambda,\theta) = \frac{\lambda^3 e^{-\lambda(e^{\theta v}-1)}}{(2+\lambda+\lambda^2)}\theta e^{\theta v}\left[e^{\theta v} + (e^{\theta v} - 1)^2\right]$

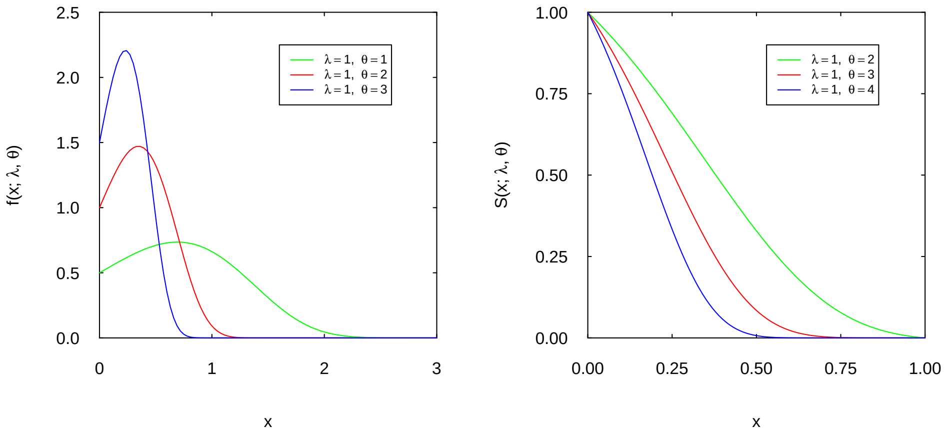

Figure 2. The pdf and survival function of Odds Lindley - Exponential distribution

2.3. *Odds OPPE - Pareto Distribution*

Suppose the baseline distribution is Pareto with parameters a and $\theta > 0$. The pdf and cdf are

$$g(v;\theta,a) = \frac{\theta a^{\theta}}{v^{\theta+1}}\ ; a < v < \infty, \theta > 0,\ \ G(v,\theta,a) = 1 - \left(\frac{a}{v}\right)^{\theta}.$$

The cdf of Odds OPPE-Pareto distribution is obtained by substituting the cdf of Pareto in (3) as follows

$$F(v;\lambda,\theta,a) \quad = \quad 1 - h(\lambda)\sum_{k=0}^{s} a_k \frac{\Gamma\left(k+1, \lambda[(\frac{v}{a})^{\theta} - 1]\right)}{\lambda^{k+1}}.$$

The pdf is given by

$$f(v;\lambda,\theta,a) \quad = \quad h(\lambda)\sum_{k=0}^{s} a_k \frac{\theta v^{\theta-1}}{a^{\theta}}\left\{\left(\frac{v}{a}\right)^{\theta} - 1\right\}^{k} e^{-\lambda[(\frac{v}{a})^{\theta}-1]}\quad ; a < v < \infty, \theta > 0, \lambda > 0.$$

A list of particular forms of pdf of Odds OPPE-Pareto distribution has been given in Table 4. Some shapes of pdf and survival function of the distribution are shown in

Table 4. Different forms of Odds OPPE - Pareto Distribution

Distribution Name	**Density Function**
Odds Exponential - Pareto[21]($s = 0,\ a_0 = 1$)	$f(v;\lambda,\theta,a) = \frac{\lambda\theta}{a^\theta} v^{\theta-1} e^{-\lambda\left\{\left(\frac{v}{a}\right)^\theta - 1\right\}}$
Odds Lindley - Pareto($s = 1,\ a_0 = 1,\ a_1 = 1$)	$f(v;\lambda,\theta,a) = \frac{\lambda^2}{(1+\lambda)} \frac{\theta v^{2\theta-1}}{a^{2\theta}} e^{-\lambda\left\{\left(\frac{v}{a}\right)^\theta - 1\right\}}$
Odds Akash - Pareto($s = 2,\ a_0 = 1,\ a_1 = 0,\ a_2 = 1$)	$f(v;\lambda,\theta,a) = \frac{\lambda^3}{2+\lambda^2} \frac{\theta v^{\theta-1}}{a^\theta} \left[1 + \left\{\left(\frac{v}{a}\right)^\theta - 1\right\}^2\right] e^{-\lambda\left\{\left(\frac{v}{a}\right)^\theta - 1\right\}}$
Odds Aradhana - Pareto($s = 2,\ a_0 = 1,\ a_1 = 2,\ a_2 = 1$)	$f(v;\lambda,\theta,a) = \frac{\lambda^3}{2+2\lambda+\lambda^2} \frac{\theta v^{\theta-1}}{a^\theta} \left(\frac{v}{a}\right)^{2\theta} e^{-\lambda\left\{\left(\frac{v}{a}\right)^\theta - 1\right\}}$
Odds Sujatha - Pareto($s = 2,\ a_0 = 1,\ a_1 = 1,\ a_2 = 1$)	$f(v;\lambda,\theta,a) = \frac{\lambda^3 e^{-\lambda\left\{\left(\frac{v}{a}\right)^\theta - 1\right\}}}{2+\lambda+\lambda^2} \frac{\theta v^{\theta-1}}{a^\theta} \left[\left(\frac{v}{a}\right)^\theta + \left\{\left(\frac{v}{a}\right)^\theta - 1\right\}^2\right]$

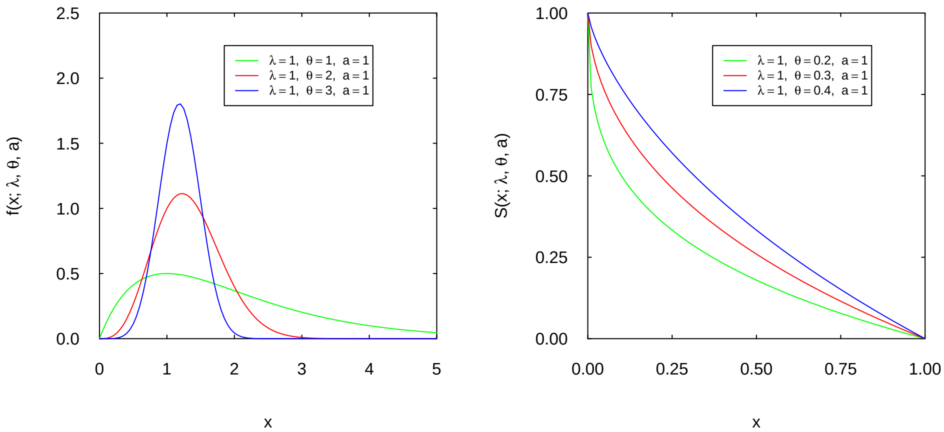

Figure 3. The pdf and survival function of Odds Lindley - Pareto distribution

Figure 3.
The survival and hazard rate functions are as follows:

$$S(v;\lambda,\theta,a) \quad = \quad h(\lambda)\sum_{k=0}^{s} a_k \frac{\Gamma\left(k+1, \lambda[\left(\frac{v}{a}\right)^\theta - 1]\right)}{\lambda^{k+1}},$$

$$r(t;\lambda,\theta,a) \quad = \quad \frac{\sum_{k=0}^{s} a_k \frac{\theta t^{\theta-1}}{a^\theta} \left\{\left(\frac{t}{a}\right)^\theta - 1\right\}^k e^{-\lambda[\left(\frac{t}{a}\right)^\theta - 1]}}{\sum_{k=0}^{s} a_k \frac{\Gamma\left(k+1, \lambda[\left(\frac{t}{a}\right)^\theta - 1]\right)}{\lambda^{k+1}}}.$$

2.4. *Odds OPPE - Burr XII Distribution*

Considering the baseline distribution as Burr (1942) with the following pdf and cdf

$$g(v;\alpha,\theta) = \alpha\theta v^{(\alpha-1)} (1+v^\alpha)^{-(\theta+1)} \quad ; v \geq 0, \alpha, \theta > 0,$$

$$G(v;\alpha,\theta) = 1 - (1+v^\alpha)^{-\theta} \quad ; v \geq 0, \alpha, \theta > 0,$$

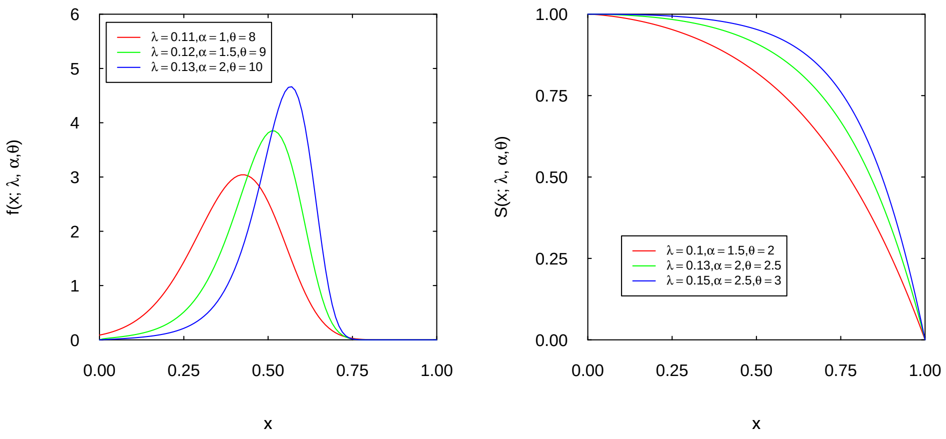

Figure 4. The pdf and survival function of Odds Lindley - Bur XII distribution

the cdf of Odds OPPE-Burr XII distribution is obtained by substituting the cdf of Burr XII in (3) as follows

$$F(v;\lambda,\alpha,\theta) \quad = \quad 1-h(\lambda)\sum_{k=0}^{s}a_k\frac{\Gamma\left(k+1,\lambda[(1+v^{\alpha})^{\theta}-1]\right)}{\lambda^{k+1}}.$$

The corresponding pdf is given by

$$\begin{aligned} f(v;\lambda,\alpha,\theta) \quad &= \quad h(\lambda)\sum_{k=0}^{s}a_k\alpha\theta v^{\alpha-1}(1+v^{\alpha})^{\theta-1}\left[(1+v^{\alpha})^{\theta}-1\right]^{k}e^{-\lambda[(1+v^{\alpha})^{\theta}-1]}; \\ & \quad\quad 0<v,\theta,\alpha<\infty,\lambda>0. \end{aligned}$$

Some particular forms of pdf of Odds OPPE-Burr XII distribution have been presented in Table 5. Some shapes of pdf and survival function of the distribution are shown in Figure 4.
The survival and hazard rate functions are given as follows:

$$S(v;\lambda,\alpha,\theta) \quad = \quad h(\lambda)\sum_{k=0}^{s}a_k\frac{\Gamma\left(k+1,\lambda[(1+v^{\alpha})^{\theta}-1]\right)}{\lambda^{k+1}},$$

$$r(t;\lambda,\alpha,\theta) \quad = \quad \frac{\sum_{k=0}^{s}a_k\alpha\theta t^{\alpha-1}(1+t^{\alpha})^{\theta-1}\left[(1+t^{\alpha})^{\theta}-1\right]^{k}e^{-\lambda[(1+t^{\alpha})^{\theta}-1]}}{\sum_{k=0}^{s}a_k\frac{\Gamma(k+1,\lambda[(1+t^{\alpha})^{\theta}-1])}{\lambda^{k+1}}}.$$

Table 5. Different forms of Odds OPPE - Burr XII Distribution

Distribution Name	**Density Function**
Odds Exponential - Burr XII($s = 0$, $a_0 = 1$)	$v/(\theta - v)$
Odds Lindley - Burr XII($s = 1$, $a_0 = 1$, $a_1 = 1$)	$f(v;\lambda,\alpha,\theta) = \frac{\lambda^2}{1+\lambda}\alpha\theta v^{\alpha-1}\left(1+v^{\alpha}\right)^{2\theta-1} e^{-\lambda[(1+v^{\alpha})^{\theta}-1]}$
Odds Akash - Burr XII($s = 2$, $a_0 = 1$, $a_1 = 0$, $a_2 = 1$)	$f(v;\lambda,\alpha,\theta) = \frac{\lambda^3\alpha\theta v^{\alpha-1}(1+v^{\alpha})^{\theta-1}}{(2+\lambda^2)e^{\lambda[(1+v^{\alpha})^{\theta}-1]}}\left[1+\left\{(1+v^{\alpha})^{\theta}-1\right\}^2\right]$
Odds Aradhana - Burr XII($s = 2$, $a_0 = 1$, $a_1 = 2$, $a_2 = 1$)	$f(v;\lambda,\alpha,\theta) = \frac{\lambda^3\alpha\theta v^{\alpha-1}}{(2+2\lambda+\lambda^2)}\left(1+v^{\alpha}\right)^{3\theta-1} e^{-\lambda[(1+v^{\alpha})^{\theta}-1]}$
Odds Sujatha - Burr XII($s = 2$, $a_0 = 1$, $a_1 = 1$, $a_2 = 1$)	$f(v;\lambda,\alpha,\theta) = \frac{\lambda^3\alpha\theta v^{\alpha-1}(1+v^{\alpha})^{\theta-1}}{(2+\lambda+\lambda^2)e^{\lambda[(1+v^{\alpha})^{\theta}-1]}}\left[(1+v^{\alpha})^{\theta}+\left\{(1+v^{\alpha})^{\theta}-1\right\}^2\right]$

3. Some Mathematical Properties

In this section, some general results of the Odds OPPE - G family are derived.

3.1. *Mathematical Expansions*

Expansion formulae of the Odds OPPE - G family, such as the pdf and cdf, are derived. The probability density function (pdf) of the Odds OPPE - G family of distributions is given by

$$\begin{aligned}
f(v;\lambda,\xi) &= h(\lambda)\sum_{k=0}^{s} a_k \frac{g(v;\xi)}{[\bar{G}(v;\xi)]^2}\left[\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right]^k e^{-\lambda\left[\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right]} \\
&= h(\lambda)\sum_{k=0}^{s} a_k \frac{g(v;\xi)}{[\bar{G}(v;\xi)]^2}\left[\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right]^k \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\lambda^i\left[\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right]^i \\
&= h(\lambda)\sum_{k=0}^{s}\sum_{i=0}^{\infty}\frac{(-1)^i a_k\lambda^i}{i!}\frac{g(v;\xi)[G(v;\xi)]^{k+i}}{[\bar{G}(v;\xi)]^{k+i+2}} \\
&= h(\lambda)\sum_{k=0}^{s}\sum_{i=0}^{\infty}\frac{(-1)^i a_k\lambda^i}{i!} g(v;\xi)[G(v;\xi)]^{k+i}[\bar{G}(v;\xi)]^{-(k+i+2)} \\
&= h(\lambda)\sum_{k=0}^{s}\sum_{i,j=0}^{\infty}\frac{(-1)^i a_k\lambda^i}{i!}\binom{i+j+k+1}{j} g(v;\xi)[G(v;\xi)]^{i+j+k} \\
&= \frac{\sum_{k=0}^{s}\sum_{i,j=0}^{\infty} w_{ijk}(\lambda)h_{i+j+k}(v;\xi)}{\sum_{k=0}^{s} w_k(\lambda)} \qquad (7)
\end{aligned}$$

where, $w_{ijk}(\lambda) = \frac{(-1)^i a_k\lambda^i}{i!}\binom{i+j+k+1}{j}$, $w_k(\lambda) = a_k\frac{\Gamma(k+1)}{\lambda^{k+1}}$, and $h_{i+j+k}(v;\xi) = g(v;\xi)[G(v;\xi)]^{i+j+k}$.

The cdf of V is given by

$$\begin{aligned}
F(v;\lambda,\xi) &= \int_0^v f(t;\lambda,\xi)dt \\
&= \frac{\sum_{k=0}^{s}\sum_{i,j=0}^{\infty} w_{ijk}(\lambda)\int_0^x g(t;\xi)[G(t;\xi)]^{i+j+k}dt}{\sum_{k=0}^{s} w_k(\lambda)} \\
&= \frac{\sum_{k=0}^{s}\sum_{i,j=0}^{\infty} w_{ijk}(\lambda)\frac{[G(v;\xi)]^{i+j+k+1}}{i+j+k+1}}{\sum_{k=0}^{s} w_k(\lambda)}. \qquad (8)
\end{aligned}$$

3.2. *Shapes of the Odds OPPE - G family of distribution*

The shapes of the density and hazard rate functions can also be described analytically. Now,

$$f(v;\lambda,\xi) = h(\lambda)\sum_{k=0}^{s} a_k \frac{g(v;\xi)}{[\bar{G}(v;\xi)]^2}\left[\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right]^k e^{-\lambda\left[\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right]}.$$

where, $h(\lambda) = \frac{1}{\sum_{k=0}^{s} a_k \frac{\Gamma(k+1)}{\lambda^{k+1}}}$.
So,

$$\ln f(v;\lambda,\xi) = \ln h(\lambda) + \ln\sum_{k=0}^{s} a_k \frac{g(v;\xi)}{[\bar{G}(v;\xi)]^2}\left[\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right]^k e^{-\lambda\left[\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right]}.$$

Now, the critical points of the Odds OPPE - G density function are the roots of the equation:

$$\frac{d}{dv}\ln f(v;\lambda,\xi) = \frac{d}{dv}\ln\left\{\sum_{k=0}^{s} a_k \frac{g(v;\xi)}{[\bar{G}(v;\xi)]^2}\left[\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right]^k e^{-\lambda\left[\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right]}\right\} = 0.$$

3.3. *Quantile function*

The quantile function, say $Q(u) = F^{-1}(u)$, of the Odds OPPE - G family is derived by inverting (3) as follows. Let

$$u = 1 - \frac{\sum_{k=0}^{s} a_k \frac{\Gamma\left(k+1,\lambda\frac{Q(u)}{1-Q(u)}\right)}{\lambda^{k+1}}}{\sum_{k=0}^{s} a_k \frac{\Gamma(k+1)}{\lambda^{k+1}}}.$$

So,

$$\sum_{k=0}^{s} a_k \frac{\Gamma\left(k+1,\lambda\frac{Q(u)}{1-Q(u)}\right)}{\lambda^{k+1}} = (1-u)\sum_{k=0}^{s} a_k \frac{\Gamma(k+1)}{\lambda^{k+1}}.$$

Taking Logarithm on both sides, the previous equation is reduced to

$$\ln\sum_{k=0}^{s} a_k \frac{\Gamma\left(k+1,\lambda\frac{Q(u)}{1-Q(u)}\right)}{\lambda^{k+1}} - \ln(1-u) - \ln\sum_{k=0}^{s} a_k \frac{\Gamma(k+1)}{\lambda^{k+1}} = 0. \quad (9)$$

By solving the nonlinear equation (9), numerically, the Odds OPPE - G family random variable X can be generated, where u has the uniform distribution on the unit interval.

3.4. *Moments*

The r^{th} moment of random variable V can be obtained using the pdf (7) as

$$\begin{aligned}\mu_r^{'} &= \int_0^\infty v^r f(v,\lambda,\xi)dv \\ &= \frac{\sum_{k=0}^{s}\sum_{i,j=0}^{\infty} w_{ijk}(\lambda)\int_0^\infty v^r h_{i+j+k}(v;\xi)dv}{\sum_{k=0}^{s} w_k(\lambda)} \\ &= \frac{\sum_{k=0}^{s}\sum_{i,j=0}^{\infty} w_{ijk}(\lambda) I_{i,j,k,r}}{\sum_{k=0}^{s} w_k(\lambda)};\ r=1,2,.... \qquad (10)\end{aligned}$$

where, $I_{i,j,k,r} = \int_0^\infty v^r h_{i+j+k}(v;\xi)dv$.
In particular, the mean and variance of Odds OPPE - G family are obtained as follows:

$$E(V) = \frac{\sum_{k=0}^{s}\sum_{i,j=0}^{\infty} w_{ijk}(\lambda) I_{i,j,k,1}}{\sum_{k=0}^{s} w_k(\lambda)},$$

$$Var(V) = \frac{\sum_{k=0}^{s}\sum_{i,j=0}^{\infty} w_{ijk}(\lambda) I_{i,j,k,2}}{\sum_{k=0}^{s} w_k(\lambda)} - \left[\frac{\sum_{k=0}^{s}\sum_{i,j=0}^{\infty} w_{ijk}(\lambda) I_{i,j,k,1}}{\sum_{k=0}^{s} w_k(\lambda)}\right]^2.$$

Additionally, measures of skewness and kurtosis of the family can be obtained, based on (10), according to the following relations

$$\gamma_1 = \frac{\mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\mu_1^{'3}}{\left(\mu_2^{'} - \mu_1^{'2}\right)^{3/2}},$$

$$\gamma_2 = \frac{\mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\mu_1^{'2} - 3\mu_1^{'4}}{\left(\mu_2^{'} - \mu_1^{'2}\right)^2}.$$

3.5. *Generating Function*

The Moment Generating function(MGF) of Odds OPPE - G family is defined as

$$M_V(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r^{'},$$

where, $\mu_r^{'}$ is the r^{th} moment about origin. Then the moment generating function of Odds OPPE - G family is obtained by using (10) as

$$M_V(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\left[\frac{\sum_{k=0}^{s}\sum_{i,j,r=0}^{\infty} w_{ijk}(\lambda) I_{i,j,k,r}}{\sum_{k=0}^{s} w_k(\lambda)}\right].$$

Characteristic Function(CF):

$$\begin{aligned}\Psi_V(t) &= E(e^{itV}) \\ &= \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\mu_r^{'} \\ &= \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\left[\frac{\sum_{k=0}^{s}\sum_{i,j,r=0}^{\infty} w_{ijk}(\lambda)I_{i,j,k,r}}{\sum_{k=0}^{s} w_k(\lambda)}\right].\end{aligned}$$

Cumulant Generating Function(CGF):

$$\begin{aligned}K_V(t) &= \ln(M_V(t)) \\ &= \ln\sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\frac{\sum_{k=0}^{s}\sum_{i,j,r=0}^{\infty} w_{ijk}(\lambda)I_{i,j,k,r}}{\sum_{k=0}^{s} w_k(\lambda)}\right].\end{aligned}$$

3.6. *Entropy*

In Descriptive Statistics, kurtosis measures the shape of the distribution of a random variable. When a random variable has a heavy-tailed distribution having all or some non-existent order moments, the kurtosis remains undetermined, and the variation of the uncertainty in that variable may be measured by entropy. A more general entropy measure was proposed by Rényi (1961). The Rényi entropy for the Odds OPPE-G distribution is defined by

$$\begin{aligned}H_R(\beta) &= \frac{1}{1-\beta}\ln\left\{\int_0^{\infty} f^{\beta}(v)dv\right\} \\ &= \frac{1}{1-\beta}\ln\left\{\frac{\int_0^{\infty}\left[\sum_{k=0}^{s}\sum_{i,j=0}^{\infty} w_{ijk}(\lambda)h_{i+j+k}(v;\xi)\right]^{\beta} dv}{\left[\sum_{k=0}^{s} w_k(\lambda)\right]^{\beta}}\right\}, \qquad (11)\end{aligned}$$

where $\beta > 0$, $\beta \neq 1$.
A special case of the Rényi entropy when $\beta \to 1$ is the Shannon entropy and is given by $E\{-\ln[f(V)]\}$.

Example 3.1. *Consider the Odds Lindley - Exponential distribution. The Rényi entropy for Odds Lindley - Exponential distribution is*

$$H_R(\beta) = -\ln\theta + \frac{\lambda\beta}{1-\beta} - \frac{\beta}{1-\beta}\ln(1+\lambda) - \frac{2\beta}{1-\beta}\ln\beta + \frac{\ln\Gamma(2\beta,\lambda\beta)}{1-\beta}$$

Shannon measure of entropy for Odds Lindley - Exponential distribution

$$\begin{aligned}H(f) = E[-\ln f(V)] &= -2\ln\lambda - \ln\theta - \lambda + \ln(1+\lambda) + \frac{e^{\lambda}}{1+\lambda}\Gamma(3,\lambda) \\ &\quad -\frac{2e^{\lambda}}{(1+\lambda)}\left[\Gamma^{(1)}(2,\lambda) - \ln\lambda.\Gamma(2,\lambda)\right]\end{aligned}$$

Example 3.2. *Consider the Odds Lindley - Pareto distribution. The Rényi entropy for Odds Lindley - Pareto distribution is*

$$\begin{aligned} H_R(\beta) &= -\frac{\ln\lambda}{\theta} - \ln\theta + \ln a + \frac{\lambda\beta}{1-\beta} - \frac{(2\beta - \frac{\beta}{\theta} + \frac{1}{\theta})}{1-\beta}\ln\beta \\ &\quad + \frac{1}{1-\beta}\ln\Gamma\left(2\beta - \frac{\beta}{\theta} + \frac{1}{\theta}, \lambda\beta\right) - \frac{\beta}{1-\beta}\ln(1+\lambda) \end{aligned}$$

Shannon measure of entropy for Odds Lindley - Pareto distribution is

$$\begin{aligned} &H(f) \\ &= E[-\ln f(V)] \\ &= -\lambda - \ln\theta + \ln a - \frac{\ln\lambda}{\theta} + \ln(1+\lambda) + \frac{e^{\lambda}}{1+\lambda}\Gamma(3,\lambda) - \frac{(2\theta-1)e^{\lambda}}{(1+\lambda)\theta}\Gamma^{(1)}(2,\lambda) \end{aligned}$$

3.7. *Order Statistics*

Order statistics are important in many sectors, including climatology, engineering, and industry, and play a significant role in real-world applications involving data from life testing studies. Let $V_{r:n}$ denote the r^{th} order statistic. The density $f_{r:n}(v)$ of the r^{th} order statistic, for $r = 1(1)n$, from independent and identically distributed random variables $V_1, V_2, ...V_n$ having the Odds OPPE-G distribution is given by

$$\begin{aligned} f_{r:n}(v) &= M.\,[F(v)]^{r-1}\,[1-F(v)]^{n-r}\,f(v) \\ &= M.\sum_{l=0}^{n-r}(-1)^l\binom{n-r}{l}[F(v)]^{r+l-1}\,f(v), \end{aligned}$$

where $M = \frac{n!}{(r-1)!(n-r)!}$
So,

$$\begin{aligned} f_{r:n}(v;\Phi) &= M.\sum_{l=0}^{n-r}(-1)^l\binom{n-r}{l}\left[1 - h(\lambda)\sum_{k=0}^{s}a_k\frac{\Gamma\left(k+1, \lambda\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right)}{\lambda^{k+1}}\right]^{r+l-1} \\ &\quad .h(\lambda)\left[\sum_{k=0}^{s}a_k\frac{g(x;\xi)}{[\bar{G}(v;\xi)]^2}\left[\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right]^k e^{-\lambda\left[\frac{G(v;\xi)}{\bar{G}(v;\xi)}\right]}\right]. \end{aligned} \tag{12}$$

3.8. *Stress-Strength Reliability*

A general measure of system performance is the probability of strength (or supply) exceeding the stress (or demand) and is known as stress-strength reliability. It is defined by $R = P(V_1 > V_2)$, where V_1 denotes the inbuilt capacity of the system to withstand, and V_2 is the load applied to that system. We derive the expression of R when V_1 and V_2 have independent Odds OPPE-G(v; λ_1, ξ) and Odds OPPE-G(v; λ_2, ξ) distributions with the same parameter vector ξ for the baseline G. The algebraic

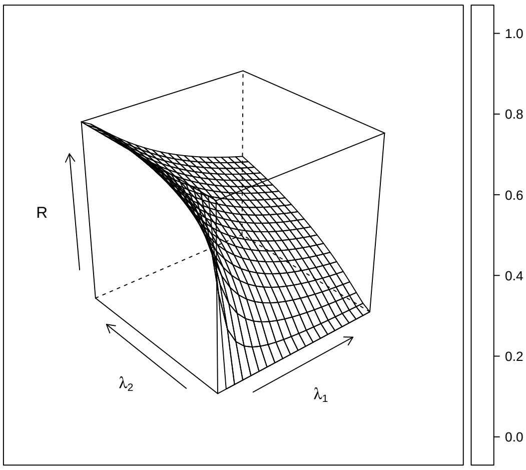

Figure 5. Stress-Strength Reliability, R for different λ_1 and λ_2 when $\theta_1 = \theta_2$, of Odds Lindley Exponential distribution

form of R is given by

$$R = \int_0^\infty f_1(v)F_2(v)dv$$

The pdf of V_1 and cdf of V_2 are obtained from equation (7) and (8) as

$$f_1(v) = \frac{\sum_{k=0}^{s}\sum_{i,j=0}^{\infty} w_{ijk}(\lambda_1)g(v;\xi)[G(v;\xi)]^{i+j+k}}{\sum_{k=0}^{s} w_k(\lambda_1)}$$

$$F_2(v) = \frac{\sum_{k=0}^{s}\sum_{i,j=0}^{\infty} w_{ijk}(\lambda_2)\frac{G(v;\xi)]^{i+j+k+1}}{i+j+k+1}}{\sum_{k=0}^{s} w_k(\lambda_2)}.$$

Hence,

$$\begin{aligned} R &= \int_0^\infty f_1(v)F_2(v)dv \\ &= \int_0^\infty \left\{ \frac{\sum_{k=0}^{s}\sum_{i,j=0}^{\infty} w_{ijk}(\lambda_1)g(v;\xi)[G(v;\xi)]^{i+j+k}}{\sum_{k=0}^{s} w_k(\lambda_1)} . \frac{\sum_{k=0}^{s}\sum_{i,j=0}^{\infty} w_{ijk}(\lambda_2)\frac{[G(v;\xi)]^{i+j+k+1}}{i+j+k+1}}{\sum_{k=0}^{s} w_k(\lambda_2)} \right\} dv \end{aligned}$$

Example 3.3. *Consider again the Odds Lindley - Exponential distribution. Let, $V_1 \sim OLED(\lambda_1, \theta_1)$ and $V_2 \sim OLED(\lambda_2, \theta_2)$ be independent random variables. Then, the*

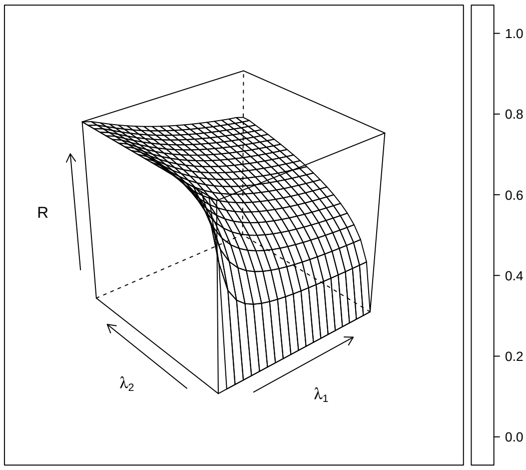

Figure 6. Stress-Strength Reliability, R for different λ_1 and λ_2 when $\theta_1 = \theta_2$, and $a_1 = a_2$ of Odds Lindley Pareto distribution

Stress-Strength Reliability is

$$\begin{aligned} R &= P(V_2 < V_1) \\ &= 1 - \frac{{\lambda_1}^2\theta_1 e^{\lambda_1+\lambda_2}}{(1+\lambda_1)(1+\lambda_2)} \int_0^\infty [1+\lambda_2 e^{\theta_2 v}] e^{2\theta_1 v} e^{-\lambda_1 e^{\theta_1 v} - \lambda_2 e^{\theta_2 v}} dv \end{aligned}$$

If $\theta_1 = \theta_2 = \theta$*, then*

$$R = 1 - \frac{{\lambda_1}^2}{(1+\lambda_1)(1+\lambda_2)} \left[\frac{1+\lambda_2}{\lambda_1+\lambda_2} + \frac{1+2\lambda_2}{(\lambda_1+\lambda_2)^2} + \frac{2\lambda_2}{(\lambda_1+\lambda_2)^3} \right]$$

The pictorial view of R for different λ_1 *and* λ_2 *is shown in Figure* 5.

Example 3.4. *Take the Odds Lindley - Pareto distribution. Let,* $V_1 \sim OLPD(\lambda_1, \theta_1, a_1)$ *and* $V_2 \sim OLPD(\lambda_2, \theta_2, a_2)$ *be independent random variables. Then, the Stress-Strength Reliability*

$$\begin{aligned} R &= P(V_2 < V_1) \\ &= 1 - \frac{{\lambda_1}^2\theta_1 e^{\lambda_1+\lambda_2}}{(1+\lambda_1)(1+\lambda_2)a_1^{2\theta_1}} \int_{a_1}^\infty [1+\lambda_2 (\frac{v}{a_2})^{\theta_2}] v^{2\theta_1 - 1} e^{-\lambda_1 (\frac{v}{a_1})^{\theta_1} - \lambda_2 (\frac{v}{a_2})^{\theta_2}} dv \end{aligned}$$

If $\theta_1 = \theta_2 = \theta$*, then*

$$R = 1 - \frac{{\lambda_1}^2 e^{\lambda_1+\lambda_2}}{(1+\lambda_1)(1+\lambda_2)a_1^{2\theta}} \left[\frac{\Gamma(2, \lambda_1 + \lambda_2 \frac{{a_1}^\theta}{{a_2}^\theta})}{(\frac{\lambda_1}{a_1^\theta} + \frac{\lambda_2}{a_2^\theta})^2} + \frac{\lambda_2}{a_2^\theta} \frac{\Gamma(3, \lambda_1 + \lambda_2 \frac{{a_1}^\theta}{{a_2}^\theta})}{(\frac{\lambda_1}{a_1^\theta} + \frac{\lambda_2}{a_2^\theta})^3} \right]$$

Also if $a_1 = a_2$*, then*

$$R = 1 - \frac{{\lambda_1}^2 e^{\lambda_1+\lambda_2}}{(1+\lambda_1)(1+\lambda_2)(\lambda_1+\lambda_2)^2}\left[\Gamma(2,\lambda_1+\lambda_2) + \frac{\lambda_2}{\lambda_1+\lambda_2}\Gamma(3,\lambda_1+\lambda_2)\right]$$

The graphical presentation of R *for different* λ_1 *and* λ_2 *is shown in Figure* 6.

3.9. *Incomplete Moments, Mean Deviations, and Lorenz and Benferroni Curves*

The r^{th} incomplete moment, say, $m_r^I(t)$, of the Odds OPPE - G Family of distributions is given by

$$m_r^I(t) = \int_0^t v^r f(v,\Phi)dv.$$

We can write from equation (7),

$$m_r^I(t) = \int_0^t v^r \left[\frac{\sum_{k=0}^{s}\sum_{i,j=0}^{\infty} w_{ijk}(\lambda) g(v;\xi)[G(v;\xi)]^{i+j+k}}{\sum_{k=0}^{s} w_k(\lambda)}\right] dv. \qquad (13)$$

Example 3.5. r^{th} *incomplete moment for Odds Lindley - Exponential distribution is*

$$\begin{aligned} m_r^I(t) &= \int_0^t v^r f(v)dv \\ &= \frac{e^{\lambda}}{(1+\lambda)\theta^r}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}(\ln\lambda)^{r-j}\left\{\Gamma^{(j)}(2,\lambda) - \Gamma^{(j)}(2,\lambda e^{\theta})\right\}. \end{aligned}$$

r^{th} *incomplete moment for Odds Lindley - Pareto distribution is*

$$\begin{aligned} m_r^I(t) &= \int_a^t v^r f(v)dv \\ &= \frac{e^{\lambda}a^r}{(1+\lambda)\lambda^{\frac{r}{\theta}}}\left[\Gamma\left(\frac{r}{\theta}+2,\lambda\right) - \Gamma\left(\frac{r}{\theta}+2,\lambda\left(\frac{t}{a}\right)^{\theta}\right)\right] \end{aligned}$$

Apart from range and s.d., mean deviation about mean, δ_1 and median, δ_2 are used as measures of spread in a population. Incomplete moments are used to define $\delta_1 = 2\mu_1' F(\mu_1') - 2m_1^I(\mu_1')$ and $\delta_2 = \mu_1' - 2m_1^I(\mu_e)$, respectively. Here, $\mu_1' = E(V)$ is to be obtained from (9) with $r = 1$, $F(\mu_1')$ is to calculated from (2), $m_1^I(\mu_1')$ is the first incomplete moment obtained from (13) with $r = 1$ and μ_e is the median of V obtained by solving (8) for $u = 0.5$.

The Lorenz and Bonferroni curves are defined by $L(p) = m_1^I(v_p)/\mu_1'$ and $B(p) = m_1^I(v_p)/(p\mu_1')$, respectively, where $v_p = F^{-1}(p)$ can be computed numerically by (8) with $u = p$. These curves are significantly used in economics, reliability, demography, insurance and medicine. For details on this aspect, we refer to Pundir et al.(2005) and the references cited therein.

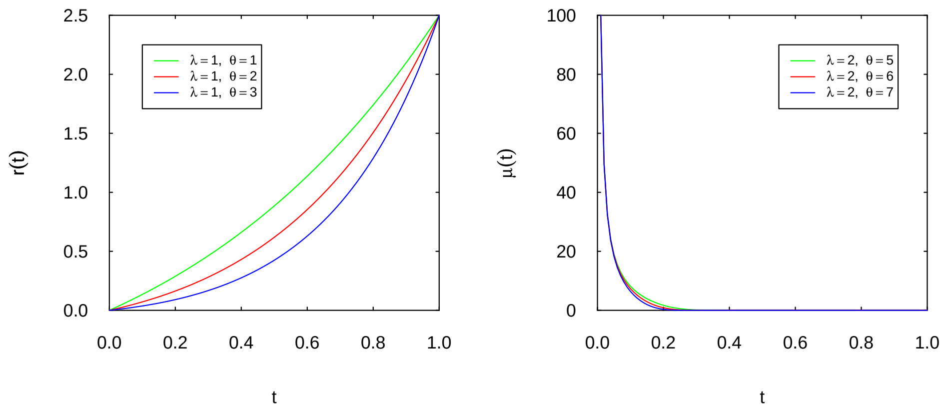

Figure 7. The Hazard Rate and Reversed Hazard Rate of Odds Lindley - Exponential distribution

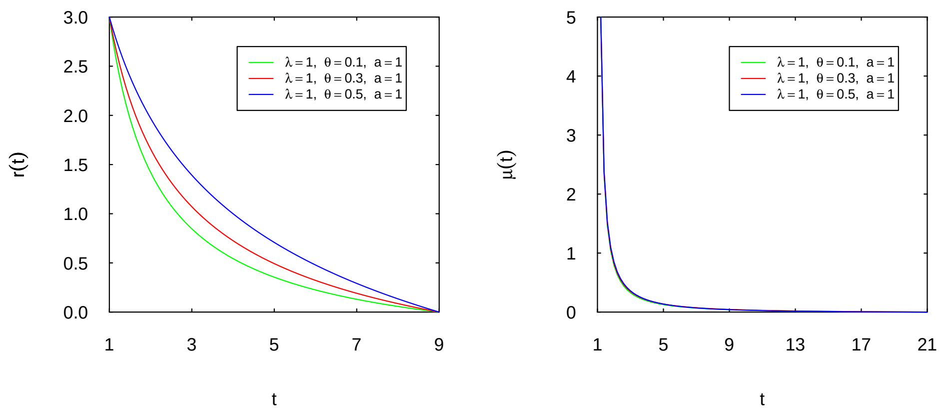

Figure 8. The Hazard Rate and Reversed Hazard Rate of Odds Lindley - Pareto distribution

3.10. *Moments of residual and reversed residual life*

The residual life function has ample applications in reliability/survival analysis, social studies, bio-medical sciences, economics, population study, the insurance industry, maintenance and product quality control, and product technology. If V is a random variable denoting the lifetime of a unit at age t, then $V_t = [V - t \mid V > t]$ is the remaining lifetime beyond that age t.
The cdf $F(v)$ is uniquely determined by the r^{th} moment of the residual life of V (for $r = 1, 2, ...$) [see, Navarro et al. (1998)], and it is given by

$$\begin{aligned} m_r(t) = E[V_t] &= \frac{1}{\bar{F}(t)} \int_t^{\infty} (v-t)^r dF(v) \\ &= \frac{1}{1-F(t)} \int_t^{\infty} (v-t)^r f(v, \Phi) dv. \end{aligned}$$

In particular, if $r = 1$, then $m_1(t)$ represents the mean residual life (MRL) function that represents the average life length for a unit that is alive at age t.

Example 3.6. *Consider the Odds Lindley - Exponential distribution. We have*

$$m_r(t) = \frac{e^{\lambda e^{\theta t}}}{1+\lambda e^{\theta t}} \sum_{j=0}^{r} \frac{(-1)^j}{\theta^j} \binom{r}{j} t^{r-j} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} (\ln \lambda)^{j-k} \, \Gamma^{(k)}(2, \lambda e^{\theta t})$$

For the MRL function,

$$m_1(t) = \frac{e^{\lambda e^{\theta t}}}{1+\lambda e^{\theta t}} \left[\frac{1}{\theta} \Gamma^{(1)}(2, \lambda e^{\theta t}) - \left(t + \frac{\ln \lambda}{\theta} \right) \Gamma(2, \lambda e^{\theta t}) \right].$$

Example 3.7. *For the Odds Lindley - Pareto distribution, we have*

$$m_r(t) = \frac{e^{\lambda (\frac{t}{a})^{\theta}}}{1+\lambda (\frac{t}{a})^{\theta}} \sum_{j=0}^{r} (-1)^j \binom{r}{j} t^{r-j} \frac{a^j}{\lambda^{\frac{j}{\theta}}} \Gamma(\frac{j}{\theta} + 2, \frac{\lambda t^{\theta}}{a^{\theta}})$$

For the MRL function,

$$\begin{aligned} m_1(t) &= \frac{e^{\lambda (\frac{t}{a})^{\theta}}}{1+\lambda (\frac{t}{a})^{\theta}} \\ &\quad \cdot \left[\frac{a}{\lambda^{\frac{1}{\theta}}} \Gamma(2+\frac{1}{\theta}, \frac{\lambda t^{\theta}}{a^{\theta}}) - t\Gamma(2, \frac{\lambda t^{\theta}}{a^{\theta}}) \right]. \end{aligned}$$

A dual notion of the residual life called the reversed residual life that takes into account the past life seems to be worth mentioning [see Di Crescenzo and Longobardi (2002)]. If the system is found to be in a failed state at a certain preassigned inspection time t, then failure relies on the past. If X be a random variable denoting the lifetime of a unit is down at age t, then $\bar{X}_t = [t - X \mid X < t]$ denotes the idle time or inactivity time or reversed residual life of the unit at age t. The reversed residual life has applications in forensic science and the insurance sector. For details, see Block et al.(1998), Chandra and Roy (2001), Maiti and Nanda (2009), and Nanda et al. (2003).

The r^{th} moment of $\bar{X}_t$ (for $r = 1,\ 2,\ ...$) is given by

$$\begin{aligned} \bar{m}_r(t) = E[\bar{V}_t] &= \frac{1}{F(t)} \int_0^t (t-v)^r dF(v) \\ &= \frac{1}{F(t)} \int_0^t (t-v)^r f(v, \Phi) dv. \end{aligned}$$

The $\bar{m}_1(t)$ represents the mean idle time or inactivity time (MIT) or reversed residual life (MRRL) function for a unit which is first observed down at age t. Some properties of MIT function have been studied by Ahmad et al. (2005) and Kayid and Ahmad (2004).

Example 3.8. *For the Odds Lindley - Exponential distribution, we have*

$$\begin{aligned} \bar{m}_r(t) &= \frac{e^{\lambda}}{1+\lambda-(1+\lambda e^{\theta v})e^{-\lambda(e^{\theta v}-1)}} \sum_{j=0}^{r} \frac{(-1)^j}{\theta^j} \binom{r}{j} t^{r-j} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} (\ln \lambda)^{j-k} \\ & \cdot \left[\gamma^{(k)}(2, \lambda e^{\theta t}) - \gamma^{(k)}(2, \lambda)\right] \end{aligned}$$

The MRRL function is given by

$$\begin{aligned} \bar{m}_1(t) &= \frac{e^{\lambda}}{1+\lambda-(1+\lambda e^{\theta t})e^{-\lambda(e^{\theta t}-1)}} \\ & \left[(t + \frac{\ln \lambda}{\theta})\{\gamma(2,\lambda) - \gamma(2, \lambda e^{\theta t})\} - \frac{1}{\theta}\{\gamma^{(1)}(2,\lambda) - \gamma^{(1)}(2, \lambda e^{\theta t})\}\right]. \end{aligned}$$

Example 3.9. *Consider the Odds Lindley - Pareto distribution and hence*

$$\begin{aligned} \bar{m}_r(t) &= \frac{e^{\lambda}}{1+\lambda-[1+\lambda(\frac{t}{a})^{\theta}]e^{-\lambda((\frac{t}{a})^{\theta}-1)}} \\ & \sum_{j=0}^{r} (-1)^j \binom{r}{j} t^{r-j} \frac{a^j}{\lambda^{\frac{j}{\theta}}} \left[\Gamma(\frac{j}{\theta}+2, \lambda) - \Gamma(\frac{j}{\theta}+2, \frac{\lambda t^{\theta}}{a^{\theta}})\right]. \end{aligned}$$

For the MRRL function,

$$\begin{aligned} \bar{m}_1(t) &= \frac{e^{\lambda}}{1+\lambda-[1+\lambda(\frac{t}{a})^{\theta}]e^{-\lambda((\frac{t}{a})^{\theta}-1)}} \\ & \left[t\{\Gamma(2,\lambda) - \Gamma(2, \frac{\lambda t^{\theta}}{a^{\theta}})\} - \frac{a}{\lambda^{\frac{1}{\theta}}}\{\Gamma(2+\frac{1}{\theta}, \lambda) - \Gamma(2+\frac{1}{\theta}, \frac{\lambda t^{\theta}}{a^{\theta}})\}\right]. \end{aligned}$$

The pictorial views of Mean Residual Life and Mean Reversed Residual Life of the Odds Lindley-Exponential and Odds Lindley-Pareto distributions have been shown in Figures 9 and 10, respectively, for illustration purposes.

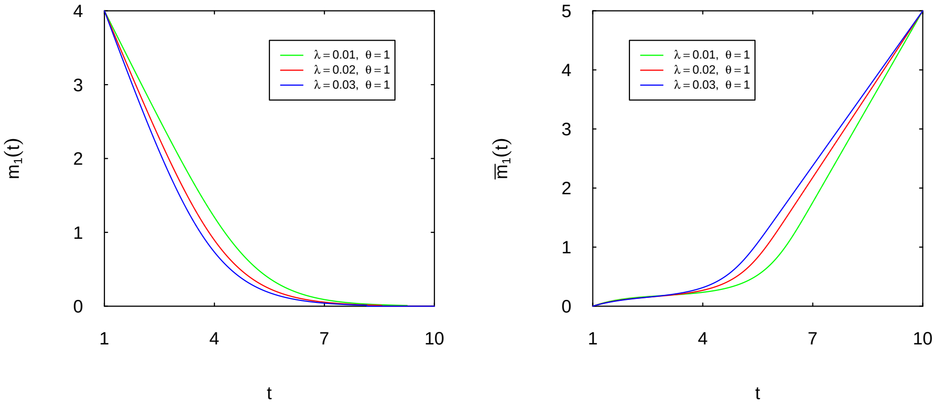

Figure 9. Mean Residual Life and Mean Reversed Residual Life of the Odds Lindley - Exponential distribution

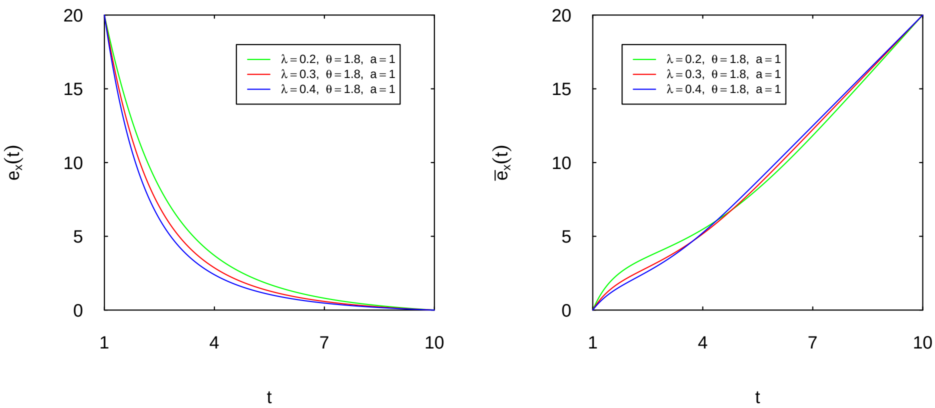

Figure 10. Mean Residual Life and Mean Reversed Residual Life of the Odds Lindley - Pareto distribution

4. Maximum Likelihood Estimation

In this section, we determine the maximum likelihood estimates(MLEs) of the model parameters of the new family of distributions from complete samples only. Let $v_1, v_2, ..., v_n$ be observed values from the Odds OPPE -G family of distributions with parameters λ and ξ. Let $\Phi = (\lambda, \xi)^T$ be the p x 1 parameter vector. The log-likelihood function for Φ is given by

$$l(\Phi) \quad = \quad n \ln h(\lambda) + \sum_{i=0}^{n} \ln \left\{ \sum_{k=0}^{s} a_k \frac{g(v;\xi)}{[\bar{G}(v;\xi)]^2} W^k(v_i;\xi) e^{-\lambda W(v_i;\xi)} \right\},$$

where $W(v_i;\xi) = G(v_i;\xi)/\bar{G}(v_i;\xi)$. The components of the score function $U(\Phi) = (U_\lambda, U_\xi)^T$ are

$$U_\lambda = \frac{n \frac{\partial}{\partial \lambda} h(\lambda)}{h(\lambda)} + \frac{\partial}{\partial \lambda} \sum_{i=0}^{n} \ln \left\{ \sum_{k=0}^{s} a_k \frac{g(v;\xi)}{[\bar{G}(v;\xi)]^2} W^k(v_i;\xi) e^{-\lambda W(v_i;\xi)} \right\}$$

and

$$U_\xi \quad = \quad \frac{\partial}{\partial \xi} \sum_{i=0}^{n} \ln \left\{ \sum_{k=0}^{s} a_k \frac{g(v;\xi)}{[\bar{G}(v;\xi)]^2} W^k(v_i;\xi) e^{-\lambda W(v_i;\xi)} \right\}.$$

Setting U_λ and U_ξ equal to zero and solving the equations simultaneously yields the MLE $\hat{\Phi} = \left(\hat{\lambda}, \hat{\xi}\right)^T$ of $\Phi = (\lambda, \xi)^T$. These equations cannot be solved analytically, and statistical software can be used to solve them numerically using iteration methods such as the Newton-Raphson-type algorithms.

5. Simulation Study

The Monte Carlo Simulation Technique is not directly applicable for generating random data from the Odds OPPE - G family of distributions since the equation F(v) = u, where u is an observation from the uniform distribution on (0,1), cannot be explicitly solved for v.

To generate random samples V_i, i = 1, 2, 3, ... n, we can use the following algorithm:

(1) Generate $U_i \sim$ Uniform(0, 1), i = 1(1)n

(2) If $\frac{\sum_{k=0}^{j-1} a_k \frac{k!}{\lambda^{k+1}}}{\sum_{k=0}^{s} a_k \frac{k!}{\lambda^{k+1}}} < U_i \leq \frac{\sum_{k=0}^{j} a_k \frac{k!}{\lambda^{k+1}}}{\sum_{k=0}^{s} a_k \frac{k!}{\lambda^{k+1}}}$, i = 1(1)s, then set $Z_i = W_i$, where $W_i \sim$ gamma(j+1, λ).

(3) If $U_i \leq \frac{a_0 \frac{1}{\lambda}}{\sum_{k=0}^{s} a_k \frac{k!}{\lambda^{k+1}}}$, then set $Z_i = Y_i$, where $Y_i \sim$ exponential(λ).

After using the odds functional form of $G(v;\xi)$, we get the ultimate random data. For Odds OPPE - Uniform model, set $V_i = \theta Z_i/(1 + Z_i)$, for Odds OPPE - Exponential model, set $V_i = \log(1 + Z_i)/\theta$, for Odds OPPE - Pareto model, set $V_i = a(1 + Z_i)^{\frac{1}{\theta}}$, and for Odds OPPE - Burr XII model, set $V_i = [(1+Z_i)^{\frac{1}{\theta}} - 1]^{\frac{1}{\alpha}}$ for generating random observation.

Here, we assume $s = 1$, $a_0 = 1$, $a_1 = 1$ to get odds Lindley- Uniform, odds Lindley-Exponential, odds Lindley- Pareto, and odds Lindley- Burr XII distribution.

A Monte Carlo simulation study was carried out 1000 (=N) times for selected values of n, λ, α, and θ.
(a) Simulation study for Odds Lindley - Uniform distribution, for first simulation, samples of sizes 20, 40, and 100 were considered, and values of λ were taken as 0.5, 1, 1.5, 3, and 6 for fixed θ=0.1. For the second simulation, samples of sizes 20, 40, and 100 were considered, and values of θ were taken as 0.1, 0.5, 1.0, 1.5, and 3 for fixed λ=0.1.
(b) Simulation study for Odds Lindley - Exponential distribution, for the first simulation, samples of sizes 20, 40, and 100 were considered, and values of λ were taken as 0.1, 0.5, 1.5, 3, and 6 for fixed θ=0.1. For the second simulation, samples of sizes 20, 40, and 100 were considered, and values of θ were taken as 0.01, 0.5, 1.0, 1.5, and 3 for fixed λ=0.1.
(c) Simulation study for Odds Lindley - Pareto distribution, samples of sizes 20, 40, and 100 were considered, and different values of λ, θ, and a were considered.
(d) Simulation study for Odds Lindley - Burr XII distribution, samples of sizes 20, 40, and 100 were considered, and different values of λ, θ, and α were considered.
The required numerical evaluations are carried out using R 3.1.1 software. The following two measures were computed:

(1) Bias of the simulated estimates $\hat{\lambda}$, $\hat{\alpha}$ and $\hat{\theta}$, for i=1, 2, 3, ...,N:
$\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda_i} - \lambda)$, $\frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha_i} - \alpha)$ and $\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta_i} - \theta)$,
(2) Mean Square Error (MSE) of the simulated estimates $\hat{\lambda}$, $\hat{\alpha}$ and $\hat{\theta}$, for i=1, 2, 3, ...,N:
$\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda_i} - \lambda)^2$, $\frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha_i} - \alpha)^2$ and $\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta_i} - \theta)^2$.

The results of the simulation study for Odds Lindley - Uniform distribution have been tabulated in Table 6. It shows that
(i) Bias and MSE decreases as n increases.
(ii) Bias and MSE increases as the values of λ increases for fixed θ=0.1.
(iii) Bias and MSE increase as the values of θ increase for fixed λ=0.1.

The results of the simulation study for Odds Lindley - Exponential distribution have been tabulated in Table 7. It shows that
(i) Bias and MSE decreases as n increases.
(ii) Bias and MSE increases as the values of λ increases for fixed θ=0.1.
(iii) Bias and MSE increase as the values of θ increase for fixed λ=0.1.

The results of the simulation study for Odds Lindley - Pareto distribution have been tabulated in Table 8. It shows that
(i) Bias and MSE decreases as n increases.
(ii) Bias and MSE increase as the values of λ and θ increase for fixed a=0.1.
(iii) Bias and MSE increase as the values of λ and a increase for fixed θ=1.

The results of the simulation study for Odds Lindley - Burr XII distribution have been tabulated in Table 9. It shows that
(i) Bias and MSE decreases as n increases.
(ii) Bias and MSE increase as the values of θ and α increase for fixed λ=0.1.
(iii) Bias and MSE increases as the values of λ and α increases for fixed θ=0.1.

Table 6. Average Bias and MSE of the estimator of $\hat{\lambda}$ and $\hat{\theta}$ for Odds Lindley - Uniform distribution

	$\hat{\lambda} = 0.5$		$\hat{\theta} = 0.1$	
n	Bias	MSE	Bias	MSE
20	-0.2776	0.0900	-0.0097	0.0003
40	-0.2754	0.0852	-0.0082	0.0002
100	-0.2646	0.0766	-0.0070	0.0001
	$\hat{\lambda} = 1$		$\hat{\theta} = 0.1$	
n	Bias	MSE	Bias	MSE
20	-0.5729	0.3665	-0.0184	0.0004
40	-0.5638	0.3661	-0.0160	0.0003
100	-0.5513	0.3301	-0.0136	0.0002
	$\hat{\lambda} = 1.5$		$\hat{\theta} = 0.1$	
n	Bias	MSE	Bias	MSE
20	-0.8772	0.8744	-0.0260	0.0007
40	-0.8749	0.8414	-0.0225	0.0005
100	-0.8422	0.7634	-0.0195	0.0004
	$\hat{\lambda} = 3$		$\hat{\theta} = 0.1$	
n	Bias	MSE	Bias	MSE
20	-1.9486	4.1142	-0.0423	0.0018
40	-1.9478	4.0112	-0.0383	0.0015
100	-1.8676	3.6703	-0.0338	0.0012
	$\hat{\lambda} = 6$		$\hat{\theta} = 0.1$	
n	Bias	MSE	Bias	MSE
20	-4.4572	20.688	-0.0608	0.0038
40	-4.4059	19.969	-0.0562	0.0032
100	-4.2437	18.447	-0.0516	0.0027
	$\hat{\lambda} = 0.1$		$\hat{\theta} = 0.1$	
n	Bias	MSE	Bias	MSE
20	-0.0561	0.0036	-0.0020	0.0003
40	-0.0557	0.0035	-0.0017	0.0002
100	-0.0536	0.0031	-0.0014	0.0001
	$\hat{\lambda} = 0.1$		$\hat{\theta} = 0.5$	
n	Bias	MSE	Bias	MSE
20	-0.0568	0.0037	-0.0097	0.0003
40	-0.0565	0.0036	-0.0085	0.0002
100	-0.0533	0.0031	-0.0071	0.0001
	$\hat{\lambda} = 0.1$		$\hat{\theta} = 1$	
n	Bias	MSE	Bias	MSE
20	-0.0556	0.0036	-0.0195	0.0004
40	-0.0555	0.0035	-0.0166	0.0003
100	-0.0529	0.0031	-0.0142	0.0002
	$\hat{\lambda} = 0.1$		$\hat{\theta} = 1.5$	
n	Bias	MSE	Bias	MSE
20	-0.0576	0.0038	-0.0298	0.0009
40	-0.0551	0.0034	-0.0253	0.0007
100	-0.0526	0.0030	-0.0213	0.0005
	$\hat{\lambda} = 0.1$		$\hat{\theta} = 3$	
n	Bias	MSE	Bias	MSE
20	-0.0563	0.0037	-0.0591	0.0037
40	-0.0555	0.0034	-0.0503	0.0027
100	-0.0529	0.0031	-0.0426	0.0019

Table 7. Average Bias and MSE of the estimator of $\hat{\lambda}$ and $\hat{\theta}$ for Odds Lindley - Exponential distribution

	$\hat{\lambda} = 0.1$		$\hat{\theta} = 0.1$	
n	Bias	MSE	Bias	MSE
20	0.0160	0.0026	0.0297	0.0012
40	0.0142	0.0017	0.0269	0.0008
100	0.0125	0.0008	0.0250	0.0007
	$\hat{\lambda} = 0.5$		$\hat{\theta} = 0.1$	
n	Bias	MSE	Bias	MSE
20	-0.1501	0.0361	0.0715	0.0057
40	-0.1414	0.0270	0.0667	0.0047
100	-0.1321	0.0203	0.0631	0.0041
	$\hat{\lambda} = 1.5$		$\hat{\theta} = 0.1$	
n	Bias	MSE	Bias	MSE
20	-0.8637	0.7744	0.1610	0.0286
40	-0.8432	0.7272	0.1489	0.0234
100	-0.8184	0.6767	0.1395	0.0199
	$\hat{\lambda} = 3$		$\hat{\theta} = 0.1$	
n	Bias	MSE	Bias	MSE
20	-2.2043	4.8961	0.2995	0.0991
40	-2.1562	4.6724	0.2688	0.0762
100	-2.1208	4.5102	0.2529	0.0657
	$\hat{\lambda} = 6$		$\hat{\theta} = 0.1$	
n	Bias	MSE	Bias	MSE
20	-5.2467	27.879	0.6196	0.4522
40	-5.0995	26.209	0.5408	0.3258
100	-4.9878	24.955	0.4782	0.2396
	$\hat{\lambda} = 0.1$		$\hat{\theta} = 0.01$	
n	Bias	MSE	Bias	MSE
20	0.0137	0.0027	0.0031	0.0003
40	0.0111	0.0014	0.0028	0.0002
100	0.0104	0.0007	0.0026	0.0001
	$\hat{\lambda} = 0.1$		$\hat{\theta} = 0.5$	
n	Bias	MSE	Bias	MSE
20	0.0141	0.0050	0.1576	0.0326
40	0.0116	0.0014	0.1388	0.0223
100	0.0053	0.0007	0.1287	0.0177
	$\hat{\lambda} = 0.1$		$\hat{\theta} = 1$	
n	Bias	MSE	Bias	MSE
20	0.0158	0.0069	0.3091	0.1306
40	0.0114	0.0026	0.2755	0.0880
100	0.0070	0.0008	0.2510	0.0673
	$\hat{\lambda} = 0.1$		$\hat{\theta} = 1.5$	
n	Bias	MSE	Bias	MSE
20	0.0147	0.0052	0.4628	0.2815
40	0.0112	0.0015	0.4147	0.1981
100	0.0080	0.0008	0.3819	0.1561
	$\hat{\lambda} = 0.1$		$\hat{\theta} = 3$	
n	Bias	MSE	Bias	MSE
20	-0.3464	0.3641	1.0259	2.2555
40	-0.3392	0.3578	0.9580	1.9045
100	-0.3315	0.3486	0.8696	1.6765

Table 8. Average Bias and MSE of the estimator of $\hat{\lambda}$, $\hat{\theta}$ and a for Odds Lindley - Pareto distribution

	$\hat{\lambda}=1$		$\hat{\theta}=1$		$\hat{a}=0.1$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	0.2650	0.6347	0.1656	0.1885	0.0092	0.0003
40	0.0660	0.2070	0.0673	0.0701	0.0048	0.0002
100	0.0370	0.0671	0.0175	0.0228	0.0020	0.0001
	$\hat{\lambda}=0.1$		$\hat{\theta}=1$		$\hat{a}=0.1$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	0.2336	0.1435	0.0833	0.0708	0.2371	0.0869
40	0.1213	0.0622	0.0432	0.0307	0.1440	0.0322
100	0.0573	0.0320	0.0147	0.0116	0.0733	0.0087
	$\hat{\lambda}=0.5$		$\hat{\theta}=2$		$\hat{a}=0.1$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	0.1601	0.2125	0.1923	0.4370	0.0121	0.0003
40	0.0948	0.1096	0.0741	0.1847	0.0067	0.0002
100	0.0346	0.0365	0.0382	0.0619	0.0028	0.0001
	$\hat{\lambda}=0.5$		$\hat{\theta}=2$		$\hat{a}=0.5$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	0.1396	0.2372	0.2277	0.4636	0.0589	0.0060
40	0.0938	0.1213	0.0898	0.1728	0.0338	0.0020
100	0.0372	0.0376	0.0327	0.0599	0.0147	0.0004
	$\hat{\lambda}=1$		$\hat{\theta}=1$		$\hat{a}=0.5$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	0.4334	0.6902	0.1434	0.1785	0.0469	0.0045
40	0.0775	0.2066	0.0558	0.0648	0.0244	0.0011
100	0.0311	0.0808	0.0268	0.0270	0.0098	0.0002

Table 9. Average Bias and MSE of the estimator of $\hat{\lambda}$, $\hat{\theta}$ and $\hat{\alpha}$ for Odds Lindley - Burr XII distribution

	$\hat{\lambda}=0.1$		$\hat{\theta}=0.1$		$\hat{\alpha}=0.1$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	-0.0030	0.0037	0.0129	0.0144	0.0222	0.0068
40	-0.0011	0.0015	0.0078	0.0072	0.0165	0.0034
100	-0.0002	0.0007	0.0046	0.0068	0.0117	0.0019
	$\hat{\lambda}=0.1$		$\hat{\theta}=0.5$		$\hat{\alpha}=0.1$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	-0.0127	0.0051	0.1157	0.5855	0.1019	0.0285
40	-0.0103	0.0024	0.1054	0.4946	0.0914	0.0222
100	-0.0068	0.0010	0.0390	0.2629	0.0747	0.0175
	$\hat{\lambda}=0.1$		$\hat{\theta}=0.5$		$\hat{\alpha}=0.5$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	-0.0133	0.0043	0.1516	0.7191	0.6022	0.8379
40	-0.0117	0.0024	0.0996	0.5498	0.5712	0.7528
100	-0.0081	0.0011	0.0494	0.3546	0.4923	0.6174
	$\hat{\lambda}=0.5$		$\hat{\theta}=0.1$		$\hat{\alpha}=0.1$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	-0.0144	0.0667	-0.0124	0.0361	0.2779	0.2553
40	-0.0061	0.0313	-0.0087	0.0135	0.2356	0.1801
100	-0.0060	0.0109	-0.0040	0.0055	0.1485	0.0997
	$\hat{\lambda}=0.5$		$\hat{\theta}=0.1$		$\hat{\alpha}=0.5$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	-0.0082	0.0632	-0.0155	0.0501	1.1588	3.7477
40	-0.0016	0.0296	-0.0118	0.0178	1.0481	2.7680
100	-0.0010	0.0125	-0.0016	0.0043	0.6658	1.5149

Table 10. Summarized results of fitting different distributions for data set 1

Distribution	Estimate of the parameters	Log-likelihood	AIC
Extended Burr XII Distribution	$\hat{c} = 2.8689, \hat{\lambda} = 0.8811$	−39.2000	82.4000
Odds Lindley Burr XII Distribution	$\hat{\lambda} = 1.5521, \hat{\theta} = 0.3949, \hat{\alpha} = 2.4772$	−38.1627	82.3255

Table 11. Summarized results of fitting different distributions for data set 2

Distribution	Estimate of the parameters	Log-likelihood	AIC
EEFr	$\hat{\alpha} = 29.5053, \hat{\beta} = 0.6415, \hat{\theta} = 0.7419, \delta = 928.9561$	−583.317	1174.634
Odds Lindley Burr XII	$\hat{\lambda} = 0.00066, \hat{\theta} = 0.62758, \hat{\alpha} = 2.53804$	−581.821	1169.643

6. Application

In this section, we fit different probability models belonging to the Odds OPPE-G family to four real data sets.

Data Set 1: The first data set consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul (Hinkley 1977). The data are: 0.77, 1.74, 0.81, 1.2, 1.95, 1.2, 0.47, 1.43, 3.37, 2.2, 3, 3.09, 1.51, 2.1, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9, 2.05. Ghosh and Bourguignon (2017) fitted this data to the Extended Burr XII distribution. We have fitted this data set with the Odds Lindley Burr XII distribution. The estimated values of the parameters were $\hat{\lambda} = 1.5521$, $\hat{\theta} = 0.3949$ and $\hat{\alpha} = 2.4772$ with Log-Likelihood=−38.1627 and AIC = 82.3255. Histogram and fitted Odds Lindley Burr XII curve to data set 1 have been shown in Figure 11.

Data Set 2: The second data set consists of the annual maximum daily precipitation (unit: mm) at Busan, Korea, for the period 1904 to 2011. The data were obtained from the Korean Meteorological Administration(2013). The data are: 24.8, 140.9, 54.1, 153.5, 47.9, 165.5, 68.5, 153.1, 254.7, 175.3, 87.6, 150.6, 147.9, 354.7, 128.5, 150.4, 119.2, 69.7, 185.1,153.4, 121.7, 99.3, 126.9, 150.1, 149.1, 143, 125.2, 97.2, 179.3,125.8, 101, 89.8, 54.6, 283.9, 94.3, 165.4, 48.3, 69.2, 147.1, 114.2, 159.4, 114.9, 58.5, 76.6, 20.7, 107.1, 244.5, 126, 122.2, 219.9, 153.2, 145.3, 101.9, 135.3, 103.1, 74.7, 174, 126, 144.9, 226.3, 96.2, 149.3, 122.3, 164.8, 188.6, 273.2, 61.2, 84.3, 130.5, 96.2, 155.8, 194.6, 92, 131, 137, 106.8, 131.6, 268.2, 124.5, 147.8, 294.6, 101.6, 103.1, 247.5, 140.2, 153.3, 91.8, 79.4, 149.2,168.6, 127.7, 332.8, 261.6, 122.9, 273.4, 178, 177, 108.5, 115,241, 76, 127.5, 190, 259.5, 301.5. The histogram shows that the data set is positively skewed. Mansoor et al.(2016) fitted this data to the Exponentiated Extended Frechet Distribution(EEFr). We have fitted this data set with the Odds Lindley Burr XII distribution. The estimated values of the parameters were $\hat{\lambda} = 0.00066$, $\hat{\theta} = 0.62758$ and $\hat{\alpha} = 2.53804$ with Log-Likelihood=−581.8215 and AIC = 1169.643. Histogram and fitted Odds Lindley Burr XII curve to data set 2 have been shown in Figure 12.

Data Set 3: The third real data set represents the survival times in weeks of 33 patients suffering from acute myelogenous leukemia. These data have been analyzed by Feigl and Zelen (1965). The data are: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43. The histogram shows that the data set is positively skewed. Mead et al.(2017) fitted this data to the Beta Exponential Frechet distribution(BExFr). We have fitted this data set with the Odds Aradhana-Pareto distribution. The estimated values of the parameters are $\lambda = 0.81667$, $\theta = 0.39864$, $a = 1$, log-likelihood =−145.8608 and AIC = 297.7216.

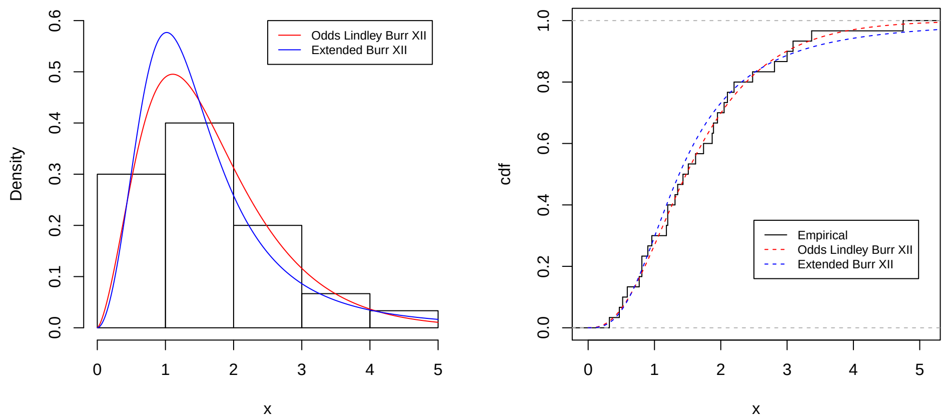

Figure 11. Plots of the estimated pdf and cdf of the Odds Lindley Bur XII model for data set 1

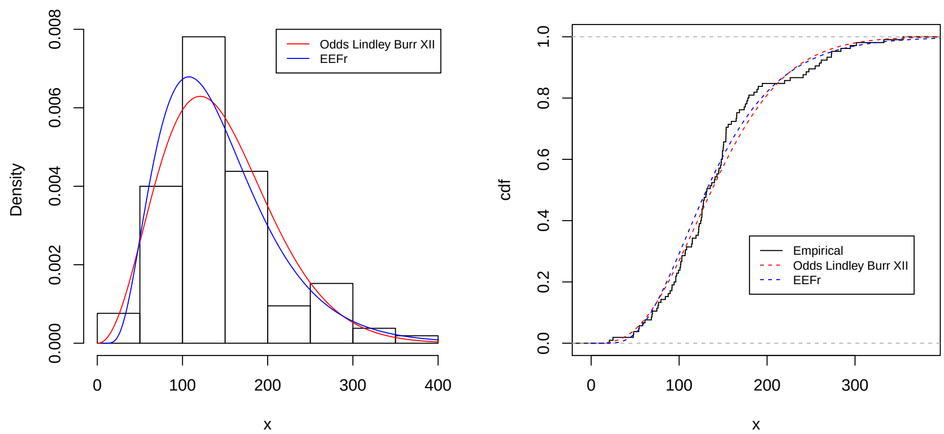

Figure 12. Plots of the estimated pdf and cdf of the Odds Lindley Burr XII model for data set 2

Table 12. Summarized results of fitting different distributions to data set 3

Distribution	Estimate of the parameters	Log-likelihood	AIC
BExFr	$\hat{\theta} = 29.588, \hat{\beta} = 0.111, \hat{a} = 21.041, \hat{b} = 19.731, \hat{\lambda} = 1.725$	-154.058	318.12
Odds Aradhana Pareto	$\hat{\lambda} = 0.8167, \hat{\theta} = 0.3986, \hat{a} = 1$	-145.861	297.72

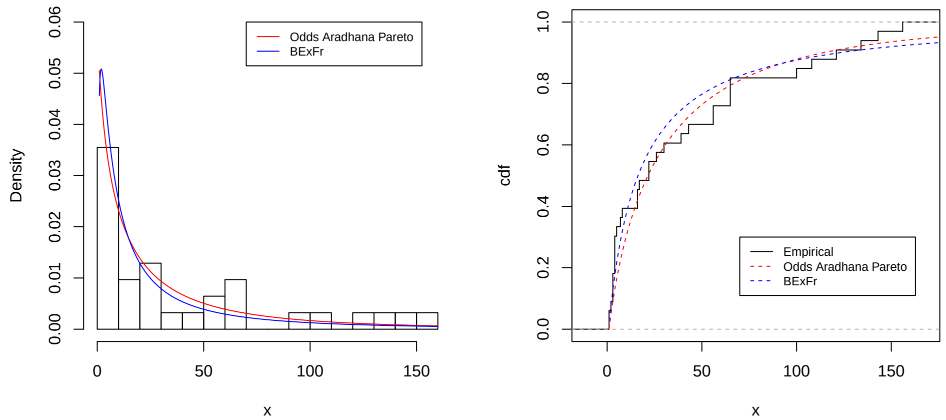

Figure 13. Plots of the estimated pdf and cdf of the Odds Aradhana Pareto model for data set 3

Histogram and fitted Odds Aradhana Pareto curve to data have been shown in Figure 13.

Data Set 4: The fourth real data set represents breaking stress of carbon fibres (Gba). The data set consists of 100 observations. The data are: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65. The histogram shows that the data set is approximately symmetric. Boikanyo et al.(2018) fitted this data to the Exponentiated Burr XII Weibull Distribution. We have fitted this data set with the Odds Lindley-Pareto distribution. The estimated values of the parameters are $\lambda = 0.0438$, $\theta = 1.9349$, $a = 0.39$, log-likelihood $=-136.8601$ and AIC $= 279.7203$. Histogram and fitted Odds Lindley-Pareto curve to data have been shown in Figure 14.

Table 13. Summarized results of fitting different distributions to data set 4

Distribution	Estimate of the parameters	Log-likelihood	AIC
EBW	$\hat{\alpha} = 0.099, \hat{\beta} = 2.5746, \hat{\delta} = 1.2187, \hat{c} = 32.602, \hat{k} = 0.0019$	-141.258	292.516
Odds Lindley Pareto	$\hat{\lambda} = 0.0438, \hat{\theta} = 1.9349, \hat{a} = 0.39$	-136.860	279.720

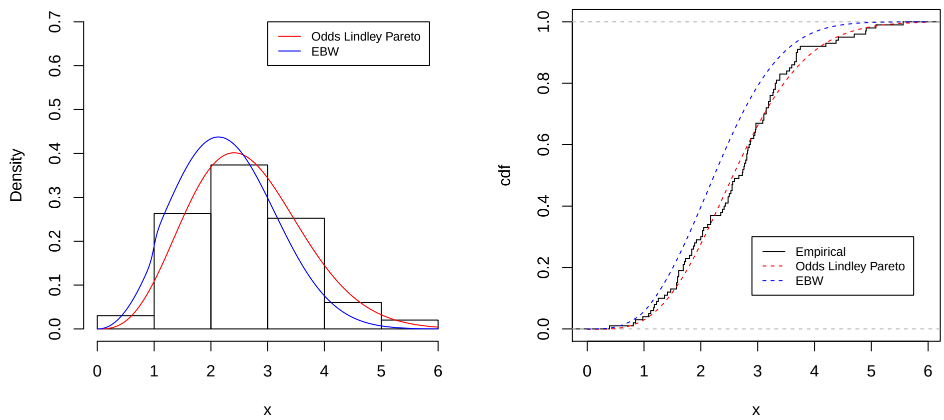

Figure 14. Plots of the estimated pdf and cdf of the Odds Lindley-Pareto model for data set 4

7. Concluding Remarks

We have introduced and studied a new generalized family of distributions called the Odds OPPE - G Family of distributions. Properties of the Odds OPPE - G Family of distributions include an expansion for the density function and expressions for the quantile function, moment generating function, ordinary moments, incomplete moments, mean deviations, Lorenz and Bonferroni curves, reliability properties including mean residual life and mean inactivity time and order statistics. The method of maximum likelihood is employed to estimate the model parameters. To illustrate the flexibility of the proposed model, four actual data sets are fitted. Compared to regular models, the discussed models provide better fitting. The article's findings are anticipated to be very helpful for professionals in various applied sciences, statistics, and probability sectors.

Acknowledgement

The authors thank the Editor-in-Chief and an anonymous referee for their comments, which have enhanced the quality of the work.

References

(1) Ahmad I A, Kayid M, Pellerey F (2005). "Further results involving the MIT order and the IMIT class". *Probability in the Engineering and Informational Sciences*, **19**, 377-395.

(2) Alzaatreh A, Lee C, Famoye F (2013). "A New Method for Generating Families of Distributions". *Metron*, **71**, 63-79.

(3) Azzalini A (1985). "A Class of Distributions which Includes the Normal Ones". *Scand J Stat.*, **12**, 171-178.

(4) Block H W, Savits T H, Singh H (1998). "The reversed hazard rate function".

Probability in Engineering and Informational Sciences, **12**, 69-90.
(5) Bourguignon M, Silva R B, Cordeiro G M (2014). "The Weibull-G Family of Probability Distributions". *Journal of Data Science*, **12**, 53-68.
(6) Burr I W (1942). "Cumulative frequency functions". *Ann. Math. Stat.*, **13**, 215-232.
(7) Chandra N K, Roy D (2001). "Some results on reversed hazard rate". *Probability in the Engineering and Informational Sciences*, **15**(1), 95-102.
(8) Cordeiro G M, Barreto-Souza (2009). "General Results for a Class of Beta G Distributions". Unpublished material.
(9) Cordeiro G M, deCastro M (2011). "A New Family of Generalized Distributions". *Journal of Statistical Computation and Simulation*, **81**, 883-893.
(10) Di Crescenzo A, Longobardi M (2002). "Entropy-based measure of uncertainty in past lifetime distributions". *Journal of Applied Probability*, **39**, 434-440.
(11) Eugene N, Lee C, Famoye F (2002). "Beta-normal Distribution and Its Applications". *Journal of Applied Probability*, **31**, 497-512.
(12) Ghitany M E, Atieh B, Nadarajah S (2008). "Lindley Distribution and Its Applications". *Mathematics and Computers in Simulation*, **78**, 493-506.
(13) Jones M C (2009). Kumaraswamy's Distribution: "A Beta-Type Distribution with Tractability Advantages". *Statistical Methodology*, **6**, 70-81.
(14) Kayid M, Ahmad I A (2004). "On the mean inactivity time ordering with reliability applications". *Probability in the Engineering and Informational Sciences*, **18**(03), 395-409.
(15) Kotz S, Lai C D, Xie M (2003). "On the Effect of Redundancy for Systems with Dependent Components". *IIE Trans*, **35**, 1103-1110.
(16) Maiti S S, Nanda A K (2009). "A loglikelihood-based shape measure of past lifetime distribution". *Calcutta Statistical Association Bulletin*, **61**, 303-320.
(17) Maiti S S, Pramanik S (2015). "Odds Generalized Exponential-Exponential Distribution". *Journal of Data Science*, **13**, 733-754.
(18) Maiti S S, Pramanik S (2016a). "Odds Generalized Exponential Uniform Distribution and its Application". *Research & Reviews: Journal of Statistics*, **5**(1), 33-45.
(19) Maiti S S, Pramanik S (2016b). "Odds Generalized Exponential-Pareto Distribution: Properties and Application". *Pakistan Journal of Statistics and Operations Research*, **12**(2), 257-279.
(20) Maiti S S, Pramanik S (2018). "Odds Xgamma – G Family of Distributions". *IAPQR Transactions*, **43**(2), 135-163.
(21) Marshall A N, Olkin I (1997). "A New Method for Adding a Parameter to a Family of Distributions with Applications to the Exponential and Weibull Families". *Biometrika*, **84**, 641-552.
(22) McDonald J B (1984). "Some Generalized Functions for the Size Distribution of Income". *Biometrika*, **52**, 647-663.
(23) Nanda A K, Singh H, Misra N, Paul P (2003). "Reliability properties of reversed residual lifetime". *Communications in Statistics-Theory and Methods*, **32**, 2031-2042.
(24) Navarro J, Franco M, Ruiz J M (1998). "Characterization through moments of the residual life and conditional spacings". *Sankhya*, **A 60**, 36-48.
(25) Pundir S, Arora S, Jain K (2005). "Bonferroni curve and the related statistical inference". *Statistics & Probability Letters*, **75**(2), 140-150.
(26) Renyi A (1961). "On Measures of Entropy and Information. In: Proceedings of the 4th Berkeley Symposium on Mathematical". *Statistics and Probability,*

University of California Press, Berkeley.

(27) Sen S, Maiti S S, Chandra N (2016). "The xgamma distribution: Statistical properties and application". *Journal of Applied Statistical Methods*, **15**(1), 774-788.

(28) Silva F G, Percontini A, Brito E D, Ramos W M, Venancio R, Cordeiro G (2017). "The Odd Lindley-G Family of Distributions". *Austrian Journal of Statistics*, **46**, 65-87.

(29) Zakerzadah, H. and Dolati, A., (2010): "Generalized Lindley distribution", Journal of Mathematical Extension, **3**(2), 13-25.

(30) Bakouch, H.S., Al-Zahrani, B.M., Al-Shomrani, A.A., Marchi, V.A., and Louzada, F. (2012): "An Extended Lindley distribution", Journal of the Korean Statistical Society, **41**, 75-85.

(31) Elbatal, I., Merovci, F., and Elgarhy, M. (2013): "A New Generalized Lindley distribution", Mathematical Theory and Modeling, **3**(13), 30-47.

(32) Shanker, R., Sharma, S.,and Shanker, R., (2013): "A two-parameter Lindley distribution for modeling waiting and survival times data", Applied Mathematics, **4**, 363-368.

(33) Ghitany, M., Al-Mutairi, D., Balakrishnan, N. and Al-Enezi, I. (2013): "Power Lindley distribution and associated inference", Computational Statistics and Data Analysis, **64**, 20-33.

(34) Singh, S.K., Singh, U. and Sharma, V.K. (2014): "The Truncated Lindley Distribution Inference and Application", Journal of Statistics Applications and Probability, **3**(2), 219-228.

(35) Abouammoh, A.M., Alshangiti, A.M. and , I.E. (2015): "A new generalized Lindley distribution". Journal of Statistical computation and simulation, **85**(18), 3662-3678.

(36) Bouchahed, L. and Zeghdoudi, H. (2018): "A new and unified approach in generalizing the Lindley's distribution with applications", Statistics in Transition, **19**(1), 61-74.

(37) Ghosh, I. and Bourguignon, M. (2017): "A New Extended Burr XII Distribution", Austrian Journal of Statistics, **46**, 33-39.

(38) M. Mansoor, M. H. Tahir, Ayman Alzaatreh, Gauss M. Cordeiro, M. Zubair and Syed S.A. Ghazali(2016): "An Extended Frechet Distribution: Properties and Applications", Journal of Data Science, **14**, 167-188.

(39) M. E. Mead, Ahmed Z. Afify, G. G. Hamedani and Ghosh, I(2017): "The Beta Exponential Frechet Distribution with Applications", Austrian Journal of Statistics, **46**, 41-63.

(40) M, Boikanyo., D, Neo., O, Broderick., and F, Adeniyi(2018): "The Exponentiated Burr XII Weibull Distribution: Model, Properties and Applications", Journal of Data Science, **16**(3), 431-462.